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THE MATHEMATICS TEACHER

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THE STRENGTH OF THE MENTAL CONNECTIONS FORMED IN ALGEBRA[1]

By EDWARD L. THORNDIKE
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As things are now, pupils lack mastery of the elements of algebra. The extent to which this is the case can be understood and appreciated best by the consideration of actual test results. The tasks shown in Table I. were the first twenty-eight of forty making a test for which from 90 to 100 minutes was allowed and which could be done by first rate algebraists in twenty-five minutes without errors save an occasional lapse.[2] Few or no complaints were made about insufficient time, and almost all the pupils attempted all of these twenty-eight tasks, and others beyond them. The schools were either private schools with excellent facilities, or public high schools in cities which rank much above the average of the country in their provision for education. In both cases the pupils would, beyond question, be superior to the average of second-year high-school pupils in general intellect and capacity for mathematics. All the pupils had studied algebra for at least one year. Most of them were continuing their study of it at the time the test was given (in October and November and December, 1921).

It does not seem an exaggeration to say that, on the whole these students of algebra had mastery of nothing whatsoever. There was literally nothing in the test that they could do with anything like one hundred per cent efficiency. If they had been asked to add $3a$ to $7a$, or to multiply $3b$ by $2b$, we might have had nearly perfect records, but that would not have meant mastery of

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[1] The studies reported in this article are made possible by a grant from the Commonwealth Fund.

[2] Four forms of the test were used, different in the concrete details of each task, but constructed on the same plan and of almost exactly equal difficulty. The results where forms B, C, and D were used may therefore be safely used to make the percentages more reliable; and this has been done.

TABLE 1.

Percentages of Wrong Responses and Failures to Twenty-eight Tasks in Ten Schools.

	School									
TASK	A	B	C	D	E	F	G	H	I	J
1. $(5a^2 + 4 - 3a^2) + (a^2 - {}^27a^2 - 5)$	11	4	10	7	23	18	28	54	46	47
2. $(-2a^3 - 10a - 4a^3) + (5a + 3a^3 - 4a)$	14	19	4	7	20	25	35	45	44	51
3. From $3a + 4b$ subtract $5a - 9b - 3c$	6	9	8	7	17	17	35	35	52	65
4. From $5a - b - 2c$ subtract $3c - 3a$	4	6	7	10	9	18	51	32	59	65
5. $8a + 8b - (3a + 6b)$	4	9	7	12	11	17	39	38	53	63
6. $(5d - e) - (7e + 2f)$	14	11	12	21	20	28	42	41	66	61
7. $7d \times 2de^2$	7	2	1	0	6	11	26	30	28	26
8. $de^2 \times d^2e$	9	9	8	2	8	24	28	38	48	37
9. $8 - 5\ (d + 2)$	10	6	24	21	20	34	29	61	80	63
10. $4e^2 + e\ (-4e - 3)$	10	6	17	14	14	30	43	60	73	70
11. $5np - 3p\ (4n + 3p)$	14	9	21	14	15	28	36	58	74	61
12. $m + \frac{8\ m^2n^2}{mn}$	39	11	31	26	20	44	72	71	75	81
13. $4m + \frac{2m^2n\ \ p^3}{mp}$	39	26	36	45	23	40	78	80	88	84
14. $\frac{m^2np}{mn} - \frac{m^2n - mp}{m}$	47	30	44	45	42	62	87	84	94	95
15. $(m^5n)\ (m^2n^3)$	3	2	9	5	5	23	39	58	52	35
16. $(2a - 7)^2$	4	6	7	14	6	13	26	34	46	49
17. If $a = 2$, and $b = 3$, what does $5a^2 - 2ab$ equal?	11	0	8	5	15	18	21	44	62	51

18. If $a = .7$, and $b = 1.2$, what does $2a^2 - 5ab$ equal?	57	45	46	50	63	64	80	78	91	91
19. If $a = 1$, $b = 2$, $c = .4$, and $d = 100$, what does $a^2 - b + cd$ equal?	44	26	36	36	32	54	71	69	90	84
20. If $a = 12$, $b = 6$, $c = 5$, $d = 3$, and $e = 1$, what does $\frac{22}{7}\,[ab + c\,(d - e)]$ equal?	59	30	42	33	37	63	71	74	93	84
21. If $d = 2$, $e = 3$, $f = 4$, what does $\dfrac{\frac{df}{d+e}}{ef}$ equal?	16	9	17	19	22	28	48	63	81	77
22. $d = \frac{ef}{g}$. What does f equal?	14	15	13	21	31	35	58	70	91	86
23. $\frac{e}{W} = \frac{r}{R}$. What does W equal?	31	17	9	21	35	28	67	80	95	91
24. $\frac{PV}{T} = \frac{P_1V_1}{T_1}$. What does V equal?	40	30	12	21	49	44	77	82	96	91
25. $4q = 7q + 5$. What does q equal?	17	19	9	17	26	33	46	63	71	63
26. $15 = 7w - 4$. What does w equal?	13	9	7	10	25	26	33	55	58	49
27. $\frac{V}{4} = a - 2$. What does V equal.	10	2	5	5	22	19	49	62	74	74
28. $\frac{6}{2-u} - \frac{11}{3-u} = 0$. What does u equal?	36	21	28	50	32	54	62	89	95	98

$3a + 7a$ or $3b \times 2b$. Complicate the situation slightly, as in $7d \times 2de^2$ or $de^2 \times d^2e$ or $4e^2 + e(-4e-3)$, which are Nos. 7, 8 and 10 of the test, and the pupils fail.

These results are supported by the findings in all tests of algebraic abilities that have been published. They have not stood out in such clear relief before, since in some of the previously given tests the pupils have been urged especially to speed, and in others the tasks have been more elaborate, complex and difficult than those which we used. Since the communities and schools which share voluntarily in educational tests and experiments tend to be far above the average in intellectual abilities and in educational wisdom and devotion, it is safe to assume that the results from Monroe, Rugg and Clark, Hotz and Douglas represent the work of superior pupils taught by superior teachers. Our results surely do.

We quote in Table 2 the facts by Hotz ('18) for pupils who had studied algebra nine months. Hotz reports ('18 p. 4) that "Evidence collected by a system of checks . . . . seems to indicate that the time allotment was ample. . . . . . In ninety-six out of two hundred of the tests submitted to nine months' students, exercises No. 19 and No. 25 in the equation and formula test were interchanged. They were then submitted to classes on a "fifty-fifty" basis. The results showed that exercise No. 25 was solved correctly about three times as often when it came last in the list, while No. 19, on the other hand, was solved more frequently when it came nineteenth on the list. Similar checks were employed in each of the other tests, with the exception of the graph test, and similar results were obtained." Since we quote the results for the first[1] and easiest sixteen out of twenty-four tasks in the addition and subtraction test, for the first and easiest sixteen tasks out of twenty-four in the muliplicaion and division test, and for the first[2] and easiest seventeen tasks out of twenty-five in the equation and formula test, there is still less danger of improper influence of insufficient time in the case of the tasks quoted here than for the tests in general.

1 Not absolutely so; one was 17th in order.

2 Not absolutely so; one was 18th in order.

TABLE 2.

Percentages of Wrong Responses and Failures of Response in the Case of Pupils Who Have Studied Algebra 9 Months. After Hotz.

ADDITION AND SUBTRACTION.

$4r + 3r + 2r =$	1	$3a^2 - 3b - (2a^2 + 3b - 4) =$	25
$2x + 3x =$	1	$5x - [4x - (3x - 1)] =$	36
$12b + 6b - 3b =$	2	$\frac{3}{4}c - \frac{3}{8}c =$	36
$2c + \frac{1}{2}c =$	5	$\frac{3x - 2}{3} + \frac{x + 4}{6} =$	45
$7x - x + 6 - 4 =$	7	$\frac{1}{a - x} - \frac{3x}{a^2 - x^2} =$	51
$3a - 4b + 5a - 2b =$	9	$\frac{r}{r + z} - \frac{r}{r - z} =$	54
$5m + (-4m) =$	13		
$20x - (10x + 5x) =$	14		
$(4r - 5t) + (s - 3r) =$	17		
$8c - (-6 + 3c) =$	24		

MULTIPLICATION AND DIVISION.

$3 \cdot 7y =$	0	$\frac{4x^4}{5} \div 2x^2 =$	15
$\frac{12n}{4} =$	1	$(2a^2 + 7a - 9)(5a - 1) =$	30
$2a \cdot 4ab^2 =$	3	$\frac{n^4 + 7n^2 - 30}{n^2 - 3} =$	24
$6c^3 \div 2c^2 =$	4	$\frac{7a}{15} \div \frac{7a^2}{20} =$	16
$\frac{2}{3}$ of $9m =$	5	$\frac{-12x^2y^2 \cdot (x - 2)}{-3x^2y^2} =$	37
$\frac{-8a^2b}{4a^2} =$	8	$\frac{m + n}{a} \cdot \frac{b}{m^2 - n^2} =$	22
$4x \cdot (-3xy^3) =$	11	$(-3xy^3)^4 =$	38
$a^3 \cdot (-3a) \cdot (-2a) =$	12		
$\frac{18m^2n - 27mn^2}{9mn} =$	20		

EQUATIONS AND FORMULAS.

$2x = 4$	0	$\frac{y}{3} = \frac{5}{2} - \frac{y}{4}$	30
$7m = 3m + 12$	2	$\frac{1}{4}(x + 5) = 5$	36
$3x + 3 = 9$	4	$\begin{cases} 3m + 7n = 34 \\ 7m + 8n = 46 \end{cases}$	30
$5a + 5 = 61 - 3a$	7	$\frac{4}{3 - x} = \frac{2}{1 + x}$	31
$7n - 12 - 3n + 4 = 0$	7	The area of a circle $= \pi r^2$ in which $r =$ radius of the circle and $\pi = 3\frac{1}{7}$ Find the area in square feet of a circle whose radius is 7 feet.	52
$10 - 11z = 4 - 8z$	10	In the formula $RM = EL$ find the the value of M	49
$\frac{2}{3}z = 6$	10		
$c - 2(3 - 4c) = 12$	10		
$\frac{1}{2}x + \frac{1}{4}x = 3$	22		
$\frac{2x}{3} = \frac{5}{8}$	20		
The area of a triangle $= \frac{1}{2}bh$, in which $b =$ length of the base, and $h =$ height of the triangle. How many square feet are there in a triangle whose base is 10 feet, and whose height is 8 feet?	31		

There is then, obviously, need for considering the psychology of the strength of the mental connections or bonds required in algebra.

We may first consider two basic propositions.

A. These pupils could have gained the abilities needed to enable them to do these simple tasks with far fewer errors. The tasks are not beyond the intellects of most of them; the trouble is not that tasks 1 to 28 contain subtleties which they cannot comprehend.

B. These pupils would profit educationally if some of the time and thought that they and their teachers have spent in other ways had been spent in enabling them to subtract $3c - 3a$ from $5a—b—2c$; multiply $2de^2$ by $7d$; find V when $\frac{PV}{T} = \frac{P_1V_1}{T_1}$, and the like. It would be better if they wholly knew what is required to master fourteen of the twenty-eight, instead of half-knowing what is required for all of them.

Probably very, very few experts in mathematics, teaching, or psychology, could be found to dispute either of these propositions. The truth of A, for all save the few pupils who are either below Stanford Mental Age 13.5 or an Army Alpha 56 (first trial), or suffer from a special mathematical disability, could probably be deduced from psychological facts. It is also demonstrated *a posteriori* by the fact that certain schools, not superior in their student personnel, do secure substantial efficiency at such tasks.

The truth of B is argued as follows:

The disciplinary value of algebra shrinks toward zero when pupils operate it so as to fail with one out of four simple tasks. The lessons of logic, precision, and economy cannot well be transferred if they have not been learned for algebra itself. The value of algebra as a tool may fall below zero when pupils are so insecure in its technique. It may be actually better for them in after life to earn money to hire somebody to do their algebra for them than to trust their own work. The value of algebra as an inspiration and enrichment becomes very dubious. One fears that children who are so much at a loss in operating with symbols and equations lack any very beneficial ideas about symbolism or the equation.

It will be understood that we are not upholding B for all algebraic knowledge, but only, for the present, for such fundamental connections or bonds as are needed for such tasks as Nos. 1 to 28 of the table.

Indeed, one of the most promising ways to secure something like 100% efficiency with certain bonds is to sacrifice others. For example, a rather long list can be made of mnemonic bonds now often formed at considerable time cost, all of which might perhaps be replaced by "Copy these formulae carefully on a card and put such a card in a pocket of every suit of clothes (dress) you own". Another long list could be made of bonds between various disguises of $a^2 - b^2$, $a^2 + 2ab + b^2$, etc., and their factors where time might be saved, these tasks being left to be done, if at all, as "originals". Other cases where bonds may be formed to only slight or even zero strength will suggest themselves.

Two objections will be made to emphasis on the strengthening of bonds by thinkers who, while admitting the validity of propositions A and B, deprecate any tendency that may sacrifice the applications of algebra and its study of relations to formal work with symbols. They will object that the formal work has already more than its fair share of attention and that we should not be interested in creating skilful, rapid algebraic computers.

We may sympathize with these objections without abandoning the view that certain bonds need to be far stronger than they now are. We could, in fact, reduce the relative amount of formal work enormously and still give more practice to the fundamental bonds than they now receive. For example, the elimination of all work with polynomial denominators, division by a polynomial, and square root and cube root of polynomials, would leave much time free for strengthening basic bonds. Moreover, it may be that interesting applications of algebra are the very best means of strengthening them.

As to creating computers, the objection states a true and important fact, but it is not an objection. We should not care much about training algebraic computers; the "practical" utility of even the simplest algebraic computation, as such, is not widespread, as is the utility of simple arithmetical computation. Algebraic compution is however much more than a practical tool,

it is also an evidence of understanding of the algebraic principles learned and an aid in learning others. Unless the pupil has mastery for such tasks as Nos. 1 to 28, he can hardly have any real appreciation of the nature of algebraic symbolism, negative numbers, exponents, equations, or the axioms used in solving them. Nor is he probably fit to follow the derivations and proofs of formulae, nor to select the formulae which fit given problems in applied algebra, nor to apply them properly when selected. More attention to the fundamental bonds will probably be profitable, entirely apart from the improvement in computation for computation's sake.

Without further debate about the importance of strengthening these fundamental bonds[1], let us consider promising means of doing it.

The first is improved and earlier understanding of the essential fact that letters represent numbers.

A pupil may think that $ab \times b = ab^2$ with the attitude "the product of two numbers times one of them—what?" and think, when he obtains the ab^2, "This is a rule that will be true of the product of any two numbers by one of them," and half-think "$cd \times d$ would be cd^2, $xy \times y$ would be xy^2." $ab \times b = ab^2$ is to him a meaningful series like "A dog has four legs". He may on the other hand, see or hear $ab \times b = ab^2$ without any "set" of his mind toward "generalized arithmetic", and without thinking of numbers, or even, in any proper sense, of anything. $ab \times b = b^2$ is then a nonsense series like "rig fan tu lo." It will then be hard to learn and to remember, and will be a dead item of memory unrelated to $cd \times d$, $x\,y \times y$, and hardly differentiated from $\frac{ab}{b}$ or $ab + b$. If the "set" or attitude of the mind toward the first hundred or so operations with literal numbers

1 By an unfortunate choice of words, it is customary to say that the basic mental connections involved in the use of the axioms, the laws of signs, exponents in multiplication and division, removal of parentheses, and the like, should be **automatic.** Automatic is used by many psychologists to mean **"unconscious", "without awareness"!** We do **not** wish them to be automatic in the sense of **without awareness.** On the contrary, it is rather an advantage for a pupil to be rather fully aware that in $\frac{a^4 x^2}{a^2 b\, x^3}$ he is cancelling, that he leaves a^2 above as the balance from a^4 and a^2, and leaves x below as the balance from x^2 and x^3 and that b must stay in too. It is surety and readiness to act, not the absence of awareness or consciousness, that we desire. **Strong, perfect, errorless, habitual, fluent,** would perhaps be better adjectives than automatic.

is permitted to become that of learning a queer game, where you pretend to add, subtract, multiply and divide letters, there is certainty that these bonds themselves will be weak, and probability that all later practice will be much less effective than it should be. As a result of their experiments in teaching, Rugg and Clark were led to provide painstakingly for full and repeated attention to the fact that *abc—xy* mean real numbers of some real objects or quantities. Work in evaluation is of great merit in this respect, as they found.

The second means of strengthening the fundamental algebraic bonds is to form and justify the habit of expecting the operations to give a trustworthy, useful result. If, by keeping the tasks within the pupils' powers, by providing them with keys and checks for use when needed, and by other means, we give them cause to trust their algebraic results as they trust their addition of 2 and 2 or their multiplication of 10 by 10, the connections will be made with distinctness, emphasis and satisfaction. Consequently, they will grow strong rapidly. If, on the other hand, the pupil thinks $ax \times 3ax^2 = 3a^2x^3$ with no sense of security, the gain from the practice will be slight. If he feels much the same when he calls $ax \times 3ax^2$ $3ax^3$ as when he calls it $3a^2x^3$, we cannot expect rapid strengthening of the latter. Unless we are skilfull some of the pupils' practice will be practice in error and much of it will be practice in insecurity.

Some of the devices which have been found helpful in arithmetic deserve trial in algebra. Such, for example, are keyed exercises wherein the pupil can learn at once whether his response is right or wrong; and practice drills wherein he acquires a specified mastery of certain bonds before proceeding to form others. Consider material like that shown below and on next page for early work in multiplication. The pupil covers the answers with a card and looks at them to verify his answers. In early stages he may verify each answer as he obtains it. Later he may write some or all before verifying any.

PRACTICE MATERIAL: A.

$3 \times 5a$	$15a$	$3 \times a$	$3a$
$4 \times 7b$	$28b$	$4 \times b$	$4b$
$5 \times 6n$	$30n$	$9 \times k$	$9k$
$7 \times 8p$	$56p$	$7 \times t$	$7t$
$8 \times 10q$	$80q$	$6 \times y$	$6y$

$2 \times 6a^2$	$12a^2$	$5 \times c^2$	$5c^2$
$3 \times 4c^2$	$12c^2$	$7 \times d^3$	$7d^3$
$5 \times 9d^2$	$45d^2$	$6 \times x^2$	$6x^2$
$7 \times 6f^2$	$42f^2$	$4 \times ay$	$4ay$
$10 \times 2m^2$	$20m^2$	$8 \times bm$	$8bm$
$5 \times 4p3$	$20p^3$	$2 \times cg^2$	$2cg^2$
$6 \times 3q^3$	$18q^3$	$5 \times ep^3$	$5ep^3$
$7 \times 5x^4$	$35x^4$	$9 \times d^2z$	$9d^2z$
$8 \times 2y^4$	$16y^4$	$2 \times adx$	$2adx$
$9 \times 3q^5$	$27q^5$	$3 \times cyz^3$	$3cyz^3$
$2 \times 4ac$	$8ac$	$2a \times 3a$	$6a^2$
$3 \times 5bd$	$15bd$	$5b \times 9b$	$45b^2$
$4 \times 7ex$	$28ex$	$6c \times 4c$	$24c^2$
$5 \times 8mt$	$40mt$	$7x \times 2x$	$14x^2$
$6 \times 2xy$	$12xy$	$9y \times 8y$	$72y^2$
$7 \times 2ac^2$	$14ac^2$	$3c \times 4ab$	$12abc$
$8 \times 3c^2k$	$24c^2k$	$8c \times 9cg$	$72c^2g$
$9 \times 2ef^2$	$18ef^2$	$5x \times 6xy$	$30x^2y$
$8 \times 4l^2m$	$32l^2m$	$3p \times 2pv$	$6p^2v$
$6 \times 5mn^2$	$30mn^2$	$4m \times 7mp$	$28m^2p$
$4 \times 3ab^2d$	$12ab^2d$	$8d \times 2ad$	$16ad^2$
$5 \times 5de^2f^2$	$25de^2f^2$	$3b \times 8ab$	$24ab^2$
$7 \times 3p^2q^2t$	$21p^2q^2t$	$2k \times 7dk$	$14dk^2$
$8 \times 2x^2yz^3$	$16x^2yz^3$	$5x \times 9cx$	$45cx^2$
$9 \times 3w^3y^2z$	$27w^3y^2z$	$4y \times 5my$	$20my^2$
$2 \times 7x$	$14x$	$3ab \times 2ax$	$6a^2bx$
$3 \times 5u^2t$	$15u^2t$	$8ck \times 4cy$	$32c^2ky$
$4 \times 4a^3$	$16a^3$	$7mx \times 6my$	$42m^2xy$
$5 \times 4mn^2p$	$20mn^2p$	$8ep \times 5kp$	$40ekp^2$
$6 \times 9axy$	$54axy$	$9mp \times 3ap$	$27amp^2$

Consider material like that shown above which the pupil uses with the directions: "Practice with these until you can give the right answers in 15 minutes". This material also may be keyed, the keys being planned for convenient use with both oral and written practice, and used so as to economize the pupil's time and encourage him to do without the key as soon as is wise.

Practice Material: B.

$6 \times 9m$	$2d\ (a - 6cd^2)$
$4 \times 7axy$	$-am\ (mn - a)$
$3ab \times 6ax$	$p^2\ (5px + s)$
$d \times d^3$	$-cx\ (x - 5y)$
$cm \times 1{\cdot}6h$	$acy\ (2y - c)$
$2x \times 1{\cdot}4$	$3y\ (n - asy)$
$x \times 8$	$bfk\ (b - 4k)$
$7 \times 5x^4$	$-bx\ (y - b)$
$an \times np^2$	$3b\ (6 - bc)$
$ad \times 4md$	$-dv^2\ (3d - 8v)$
$ay \times 3x$	$-b^2\ (-b + y)$
$4ch \times cfh^2$	$4an^2\ (cn + 2a)$
$5df \times 3np$	$e\ (p - 8bc)$
$2ady \times 4nd^2y$	$-d^2x\ (11 - 2ax)$
$8 \times 2x^2y^2$	$4y\ (x + c)$

$9 \times 3ef^2$	$5m\ (x - am^2)$
$x \times x^3$	$-bdy\ (y - 3)$
$2cv \times 3v$	$ax^2\ (dx + 12)$
$ab \times cx$	$f\ (x - 16)$
$7 \times a^2z$	$-cp\ (y - ap)$
$5 \times 4p^3$	$-4p\ (p + 4)$
$3 \times m$	$lms\ (4 - 2l)$
$2y^2 \times y$	$-mn\ (pm - n)$
$4dw \times 5d$	$9n\ (-a - 4n)$
$-3n \times 5p$	$ay^2\ (r + 5)$
$5b \times 6b$	$2q^2\ (3q - 5)$
$3x \times 8cx$	$-a^2d^2\ (a + ab)$
$4b \times 7ab$	$m^2p\ (mp - 8)$
$7d \times 2de^2$	$-8n^3\ (b - an)$
$-pw^2 \times p^2w$	$e^2y\ (y - 3)$
$5 \times 4mn^2p$	$5x\ (7 + akx)$
$6bkx \times 3ky$	$7n^2\ (3d - 2b)$
$2dy \times 4ady$	$-x^2y^2\ (x - 3dy)$
$b \times x$	$6a\ (9 + b)$
$p \times y^2$	$-ps\ (p - 8s)$

If the attention of the class is held, rapid oral exercises are useful in algebra as in arithmetic, having the merit that a wrong response suffers immediate correction.

The third means is by infusing the process of learning with interest, so that the pupils care about obtaining right answers. Drills can probably be devised[1] that will be as suitable in the bonds formed and much more attractive than those on pages 325 and 326.

Group competition and competition by individuals each with his own past record will be found useful. The teaching of algebraic computation as a means of solving for any one of the elements of a formula and deriving new formulae from those already known will show the utility of the computations, and may thereby increase interest. Nunn's treatment should be studied from this point of view, since he has used brilliant ingenuity and much care in introducing computing as a means to "changing the subject" of a formula.[2]

A fourth means is the provision of aids to bridge the transition from learning A and B and C and D to learning to oper-

[1] Further illustrations of this sort of work, will be found in recent text books, for example, on pages 47, 176, 206, 254, 269, 279, 280, 305, and 309 of Rugg and Clark "Fundamentals of High School Mathematics".

[2] It is however a question whether the formulae of science and engineering are very much more interesting to pupils than the a's and b's and x's, and whether changing the subject of a formula and deriving new formulae from a given formula are much more real issues to them than finding sums, differences, products, and quotients. Nunn's procedure is correct and means some gain, but we should not expect too much from it.

ate A and B together, and C and D together, and later A, B, C and D all together. Thus a pupil learns to find any product of the form $a \times b$,[1] and any product of the form $ax \times by$, and any product of the form $x^n \times x^m$, and learns that $+ \times +$ gives $+$, $- \times -$ gives $+$, and $+ \times -$ or $- \times +$ gives $-$.

To multiply $3^p \times 4^{qr}$, he has to use the first two in cooperation; to multiply $(p^2)\ (-p^3)$ he has to use the last two in cooperation; to multiply $(2^c mx) \times (-03m^2px^3)$ he has to use all four (and in fact certain other bonds as well) in the right cooperative arrangement.

The organization and cooperative use of habits needs guidance as truly as their separate formation. Gradation of the tasks and keyed exercises will help to prevent practice in error and blundering. Surveying the results with respect to signs, coefficients, letters and exponents, may help. The more elaborate the selection, arrangement and relations of the habits are, the more profit there will be from checking. Pupils who are confused and react in a hit or miss way may be aided by being led to state just what they plan to do and why they plan to do it.

The most obvious means of improvement we have not yet mentioned, namely, a general increase in the amount of practice on computation. We have not mentioned it because it is doubtful whether a general increase in the kind of practice now given is an economical means of securing mastery. We do not, for example, know that the use of text-books in which the general computation is reduced enormously, results in weaker fundamental bonds. The quality of the practice is certainly the thing for science to improve. Anybody can increase its general amount. Certain inequalities and special insufficiencies should, however, receive attention.

Finally it is obvious that any improvements made in the conditions and methods of learning will tend to secure greater strength of these bonds, other things being equal. There is a positive correlation amongst schools between mastery of them and ability with more elaborate calculations and with problem solving.

[1] Letting a, b, and c represent any numerals, and letting x, y and z represent any literal factors expressed by single letters.

So much for the bonds that need to be made stronger than they are made now. Consider bonds that may well be left weaker than they are now. We have first any bonds that are useful only for abilities which have been recommended for discard. They need only zero strength. Next we have such specific memory bonds as those for the formulae of arithmetic and geometric progressions and the binomial theorem. Pupils might perhaps gain by being permitted to look these up in the book or by being given time to derive them instead of being required to learn them as now. They are evidently hard to remember; for it is a regular procedure for pupils who take college entrance examinations to study the formulae just before the examination, and write them out on the question paper as soon as they receive it, before even looking to see which it calls for. Their teachers train them to do this. If a pupil really understands them, however, it would seem that he ought to be able to remember or re-derive them after a reasonable amount of practice in applying them.

In general, in a course in Algebra such as would embody the recommendations so far made in this article, there are not many bonds formed, that are not worth forming to a strength of say, Right 99 times out of 100, when operating along with other bonds in the ordinary applications of algebra.

One very special case remains—that of crutches, or connections which are formed for temporary use only, to give way later to others. Such are: writing 1 as coefficient, writing a parenthesis around a polynomial which is a numerator or denominator or under a radical sign, and writing 1 as exponent. Such crutches are very rarely advocated by authors of text books or courses of study and are not much used by teachers of algebra.

The general principle is to avoid them except for reasons of weight, in accord with the general psychological maxim, "Other things being equal form a connection in ways in which it is to be used." When there do seem to be reasons of weight the bad consequences of the use of crutches can be reduced by attaching the standard procedure to one mental set or attitude,

and the provisional "crutch" procedure to a clearly differentiated set. For example the pupils may be given work in this form:

In this column you may change a to a^1, b to b^1, c to c^1, *ets.*, to help you to remember that when no exponent is printed the exponent 1 is understood.

$a^{\frac{1}{2}}\ (a^{\frac{1}{3}} + a)$

$b^{\frac{1}{2}}\ (a + a^{\frac{3}{4}} + b^{\frac{3}{2}})$

$c^2\ (c^{\frac{1}{2}} - c + c^{\frac{3}{2}})$

$d\ (\sqrt{d} - \sqrt{cd})$

$(e + e^{\frac{1}{2}})\ (e - e^{\frac{1}{2}})$

etc.

In doing the work of this column, remember that when no exponent is printed, the exponent 1 is understood.

$a\ (a^{\frac{3}{4}} = a)$

$b^{\frac{1}{2}}\ (a^{\frac{1}{2}}b + ab^{\frac{1}{2}} + b)$

$(c^{\frac{1}{2}} + d)\ (c + d^{\frac{1}{2}})$

$e^{\frac{3}{4}}\ (e^{\frac{1}{2}} - e^{\frac{1}{4}} + e)$

etc.

In some cases where a certain procedure eventually gives way to another the former should still be maintained at a substantial strength, because of its value as a part of the pupil's total system of algebraic abilities and as an insurance against rote learning and other calamaties. Such, for example, are the first applications of the axioms in the arrangement of equations for solving. Adding .... to both sides and subtracting .... from both sides do give way to "transposing" but they should not be permitted to starve for lack of exercise thereafter. It is true that we do not wish a pupil, after attaining $2p + 4 = p + 6$ or $10p=100$, to think laboriously, "I will subtract p from both sides and 4 from both sides," and "I will divide both by 10." On the other hand we do wish him to retain the axiom bonds strong for use when needed. The use of equations which result in $14.5p = 92.6$ and the like will serve this and other useful ends.

There is some evidence that teachers of algebra let the axiom bonds weaken from disuse too much and too soon. For example, many pupils have no clear and sure ideas of why the common denominators vanish when an equation is "cleared of fractions" and do not vanish when fractions are added or subtracted. In fact if, after the training in clearing of fractions, tasks in adding fractions are assigned, a considerable percentage of pupils discard the denominators there. In many respects it would be profitable to teach pupils at the beginning

to clear equations of fractions gradually by multiplying by the "largest" denominator first and then by the largest that remained and so on. As a general procedure for after-life this is perhaps better than finding the least common denominator, reducing all terms to it, and then letting all denominators disappear. It is easier to remember, and nearly or quite as economical of time for the sorts of operations life offers. As a procedure for school use it has the merit of reinforcing the fundamental knowledge of the equation and the use of the axioms. If pupils later learn to obtain the least common denominator and operate accordingly they will be less likely to learn it as an unreasoning routine.

Teachers sometimes treat what we have called the regular procedure almost as a crutch assisting the pupil to mastery of a short cut which replaces it. For example, a pupil would probably be scorned for multiplying out $(2m + 7)(2m + 7)$ instead of applying the $a^2 + 2ab + b^2$ formula, or for writing $(2p - q) - (3p + q)$ in column form and subtracting instead of changing signs and collecting terms, if he did either after the short-cut had been learned. He might be scorned for dividing $m^6n^3 - 27p^3$ by $m^2n - 3p$ instead of writing the result directly with the aid of $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, or for not transposing two terms from each side in one step.

In view of the very low degree of strength of the fundamental bonds, it seems unwise to abandon them so soon. Agility with algebraic manipulations is of value chiefly as a symptom of understanding of literal and negative numbers, formulae, equations, and the laws of generalized arithmetic. Mastery of the regular operations usually teaches these lessons better than facility with short cuts, and the short cuts themselves are most instructive when based on mastery of the regular operations.

FUNDAMENTAL PRINCIPLES OF ALGEBRA

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An eminent mathematician has said recently that of all the high school subjects, algebra has the least and geometry the most educational value. No study in the high school course leaves a more hazy impression on the mind of the average high school student as to its purpose and value than does algebra. The student may put in hours of hard work; he may acquire some skill in performing algebraic operations (to him a highly mechanical accomplishment); he may be able to solve a fairly large number of the problems; he may quote verbatim many definitions, rules and principles; but, when asked what algebra is "all about," what the letters mean, and whether or not there is any "point" or advantage to his accomplishments, the pupil is "at sea." In talking with students, I find that the work done by them, in many cases, is quite purposeless and meaningless. To many the algebra work is done from day to day because it is a task assigned, a sort of daily grind that they must go through, using as their guide-posts the type examples worked out in the algebra texts, or explained by the teacher in the assignment of the lesson.

For some weeks, at the beginning of the present school year, I made it a daily practice to copy on a sheet of paper erroneous statements made in the written work of students in my algebra class. These students had had one year of algebra in the high school and were reviewing in preparation for more advanced work. Samples of the type errors that I noted are:

(1) $x^2 \div x = 1^2$ (3) $(a^2)^3 = a^5$

(2) $\dfrac{a^3 + b^3}{a + b} = a^2 + b^2$ (4) $\dfrac{\cancel{a^2}}{\cancel{a^2} + ab} = ab$

We were reviewing the laws of exponents. In a test I gave the following question: Write illustrations of three laws of exponents, using numbers of arithmetic. You may be interested in some of the responses: (1) $6^2 + 6^3 = 6^5$; (2) $9^4 \div 3^2 = 3^2$; (3) $(4^2)^3 = 4^5$. In speaking of such blunders, one should not omit some which, I am sure, are familiar to all of you:

(1) $\dfrac{\cancel{x} + 1}{\cancel{x}} = 1$ (2) $\dfrac{1}{a} + \dfrac{5}{a} = \dfrac{6}{2a}$ (3) $\sqrt{a^2 + b^2} = a + b$

Pupils will say $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$, when they would never think of saying $\frac{1}{3} + \frac{1}{4} = \frac{1}{7}$. Moreover, they will say $(a + b)^2 = a^2 + b^2$, when they would not say

$$(3 + 4)^2 = 9 + 16, \text{ or } 25.$$

Now my feeling is that the types of errors which I have noted are not peculiarly characteristic of algebra, as taught in any one school. I dare say that many of you are experiencing, or have experienced, similar difficulties. There is something wrong. Where does the trouble lie? Is it to be found in the stupidity of the pupils, or in the difficulty and intangibleness of algebra itself, or does the difficulty come from a lack of emphasis on the outstanding or fundamental notions, which we, as high school teachers, should have in mind, when teaching the subject? I am going to assume that the latter is the principal source of trouble. The question that we have before us then is: What are these fundamental notions that a teacher of high school algebra should have in mind?

Let us imagine that we have here before us a cylindrical vessel the volume of which is to be found. I cut a paper the same size as the bottom and divide it into inch squares. Making due allowance for incomplete squares, I count and find that there are very nearly 22 inch squares in the paper. We shall say that there are exactly 22. If, now, I place in the cylindrical vessel a layer of clay 1 inch thick and just large enough to cover the bottom, I shall have 1 cubic inch of clay standing on each square inch in the bottom. There will be, evidently, 22 cubic inches in the layer altogether. This cylinder is 7 inches high. I pack 7 layers of clay 1 inch thick on top of one another, and the total amount of clay that the cylinder holds is 7 x 22 cubic inches, or 154 cubic inches. If there were, in the base of the cylinder, 17 or 25 or any other number of square inches, and, in the height, 5 or 11 or any other number of linear inches, it is clear that I could calculate the amount of clay the cylinder will hold in the same way, namely, 5×17 cu. in., 11×25 cu. in., and so on. Consequently, we have the following rule for finding the volume of any cylinder: The volume of a cylinder is found by multi-

plying the number of square inches in its base by the number of inches in its height. If the area of the base had been fractional, say 27.1 square inches, each of the layers of clay would have contained 27.1 cubic inches, and, if the height had been 10.4 inches, I could have packed into the cylindrical vessel 10 slabs of clay 1 inch thick and another thinner layer .4 of an inch thick. Then, extending the definition of multiplication so as to include fractional, as well as integral numbers, I could multiply as before, and the rule for finding the volume of the cylinder still holds.

But, suppose it should be necessary to write this rule in a handbook for reference. It would, evidently, take too long and take too much room to write every word as follows: The volume of a cylinder is found by multiplying the number of square inches in the area of its base by the number of inches in its height. There would be no difficulty in knowing the meaning, if one should use abbreviations and write:

$$\text{Vol. of cyl.} = \text{base} \times \text{height}.$$

There are a great many people who constantly need to make use of notes or memoranda of this kind. They are such people as engineers, who need rules for finding the weights that wooden beams and steel bars will support, and shipbuilders and architects, as well as electricians, sailors and military engineers, who use rules for finding the charge of powder needed to blow a breach in a wall of given thickness. Some of the rules they use would take up too much room in their hand-books, even if they should shorten them, as we have the rule for the volume of a cylinder, by using the abbreviation vol. for volume and ht. for height. They find it necessary to use a kind of shorthand for writing their notes. The principle they go by is to use only one letter for a word, such as "height," or a group of words, such as "the volume of a cylinder." An engineer would write the above rule as $V = B \times h$. So far as possible, the letters used are such that they suggest the words for which they stand, V for volume, B for base, and h for height. If it is agreed that two numbers are to be multiplied, when the letters referring to them are written side by side, the rule would be written $v = bh$. That is, the formula $v = bh$ is merely the

shorthand way of writing the rule, the volume of a cylinder is found by multiplying the number of square inches in its base by the number of inches in its height.

The question which I proposed for discussion is: What are the fundamental notions of algebra? So far, it seems that I have been talking about a rule in arithmetic. But in learning this rule in arithmetic, the pupil is beginning the study of algebra. Let us see if I can make clearer what I have in mind. In arithmetic the pupil learns that by the area of a figure is meant the number of square inches, or square units that will cover it, and that the area of a rectangle 7 inches long and 5 inches wide may be found by covering its surface with inch squares, so arranged that they appear in 5 rows each containing 7 inch squares. It is clear to him that the area in question is 5 times 7 square inches. So far, this example is entirely in arithmetic. But let the pupil get away from the particular numbers 5 and 7 and their use in finding the area of the particular rectangle, and try to analyze the process by which he finds the area of any rectangle. The essence of the process is the multiplication of the length of the rectangle by the width. Just at the moment the pupil makes this analysis, he has begun the study of algebra. Even so, the moment a pupil sees that the volume of a cylinder, no matter whether its base is 17, 22, or 25 square inches, or its altitude 5, 7, or 10.4 inches, is equal to its base multiplied by its height, that pupil has crossed the boundary that separates arithmetic from algebra. As stated by Nunn in his *Teaching of Algebra,* "The most fundamental element in algebra is analysis."

Arithmetic and algebra are very closely related. One studies algebra in arithmetic and arithmetic in algebra. It is not an easy matter to define algebra. Neither is it an easy matter to draw a definite line between arithmetic and algebra. In fact, it is impossible to separate two sciences so closely related. For us, the line of demarcation is not an important consideration. The important thing for us, as teachers of algebra, is to realize that the difference between arithmetic and algebra is not so much a difference in the thing studied, as a difference in the way one thinks of the thing studied, whether, as in arithmetic, the manipulation of particular numbers to get a particular

numerical result, or, as in algebra, the process involved in the manipulation to get a definite expression for this process.

There are many types of problems in arithmetic, the analysis of which, yields a rule for solving every problem of the type. In arithmetic, the pupil finds the area of a rectangle, the area of a square and of a circle, as well as the volume of a rectangular solid, of a cube, and of a cylinder. He also solves problems in finding the surface and volume of a prism, and of a pyramid, as well as of a cylinder, a cone and a sphere. In percentage problems the pupil finds the percentage by multiplying the base by the rate per cent; in interest problems, he finds the interest by multiplying the principal by the rate and this product by the time expressed in years. All of these and many more are merely examples of types of problems in arithmetic, which, when analyzed, give rise to definite rules. They are types of problems which may be used to decided advantage in beginning the study of algebra, when it is so necessary that the pupil put definite content into the symbolic expressions that he is learning to manipulate. Let us then set down, as the first fundamental element for the teacher of high school algebra to emphasize, this idea of analysis, the bringing to light of the essential process concealed in the numerical garb of arithmetical problems.

For the study of algebra, as we have seen it thus far, the essential element is the power that every pupil has, at least to some extent, of seeing the process involved in a type of arithmetical problem. A sufficient command of English to state the process, once it has been discovered, is presupposed. But, as we saw in studying the derivation of the formula for the volume of a cylinder, one cannot get on very well in writing the rules that one learns and needs without some kind of abbreviations or symbols for expressing these rules. To write $v = bh$ is a simple to gravity. In fact, without the aid of symbols, many of the matter in comparison with writing the long rule for finding the volume of a cylinder. $d = \frac{1}{2} gt^2$ requires much less time for writing and much less space than the statement: The distance through which a body falls from rest equals $\frac{1}{2}$ the product of the square of the time of falling multiplied by the constant due to gravity. In fact, without the aid of symbols, many of the rules that have been discovered in mathematics could scarcely

be made usable. One writes $(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2} a^{n-2}b^2 + \cdot \cdot \cdot$, but when the attempt is made to express in words the rule of which this is the symbolic expression, it is found very lengthy indeed, and almost useless for practical purposes.

Nunn in his book entitled "Exercises in Algebra (including Trigonometry)" gives interesting lists of exercises, for practice in expressing rules and principles in the shorthand of algebra, as well as exercises to give skill in writing formulae for solving certain types of arithmetic problem. With your permission I shall read and discuss a few of these exereises.

(1) Using the shorthand of algebra, write the rule for calculating the cost of a number of things, when you know the price of each. Stated in words the rule is: The cost is obtained by multiplying the price of each thing by the number of things. In the symbols of algebra, $C = np$.

(2) Write the rule, in the shorthand of algebra, for calculating the cost (C) of a certain number of things (N), when you know how much (c) another number of the same kind of things (n) costs. Clearly, that is: $C = N\frac{c}{n}$.

(3) A green-grocer buys a certain number of oranges at a certain price per dozen and sells them at so much each. Write a formula for his profit after selling a certain number. Of course, the profit equals the difference between the selling price and cost of each multiplied by the number of oranges sold. Hence, the formula is: $P = N(s - \frac{c}{12})$.

(4) Using the shorthand of algebra, write the rule for finding the number of spoonfuls of tea required for a certain number of persons according to the receipt, "One spoonful of tea for each person and one for the pot."

(5) The time (t) in hours for cooking a joint of beef of given weight (w) is given by the rule, "Allow a quarter of an hour for every pound and 20 minutes over." Write the rule in algebraic symbols.

(6) Write the formula for finding the weight of a bag containing 37 marbles, given the weight of a bag (b) and the weight of a single marble (m). The formula is: $w = b + 37m$.

(7) Write the formula in exercise 6 for any number of marbles (n). This gives $w = b + nm$.

(8) The length of shelf that would be occupied by 8 books of a certain thickness, followed by 5 books of another thickness is given by a formula. Write the formula. We have $l = 8t_1 + 5t_2$.

(9) Write the formula for the length of shelf that would be occupied by two sets of volumes, given the number and thickness of the volumes in each set. The formula is $l = n_1t_1 + n_2t_2$.

Now, such exercises as these are a revelation to high school pupils, who have never stopped to think what the symbols of algebra really mean. I recommend them for your use in algebra classes.

The history of algebra shows that the growth of a concise symbolism has closely paralleled the development of the science. It may be of help in appreciating the struggle that pupils have in understanding the meaning of the symbols of algebra, to mention the periods through which the world has passed in developing the present method of writing the equation, and to give an illustration, under each period, of the method used.

Three periods are rather clearly marked. The first is the rhetorical period. In the algebra of Mohammed-ben-Musa, written during the rhetorical period, is found the quadratic equation $x^2 + 10x = 39$ written as follows: "One square and ten roots of the same amount to thirty-nine." That is, equations are written entirely in words and no abbreviations or symbols are used. To Mohammed-ben-Musa the equation which I have given meant: What is the square which increased by 10 of its roots makes 39? Incidentally, this may throw some light on the reason for the expression, root of an equation.

The second period in the development of the symbolism of algebra is known as the syncopated period. The words used in writing equations were abbreviated, somewhat as the words were abbreviated in the rule for finding the volume of a cylinder. Diophantus, an Alexandrian, living during this second period, perhaps in the first half of the fourth century, wrote a work in which he represented the unknown quantity in the equation

by the final letter, *s*, of the word used for unknown quantity. He used a letter which looks somewhat like the letter i, for the word equals, it being the final letter of the word.

The third period, which I shall mention, is the period of symbolic algebra. Vieta has the honor of founding symbolic algebra, much the same as we have it today. Before his time (about 1600) unknown quantities were represented by letters, but the powerful influence of using letters for known quantities as well, had its beginning with Vieta. Algebra became a search for operations to be performed, rather than a search for particular values, that is, the idea of function was already beginning to enter the science.

Now, when it took the world until about 1600 to develop the present symbols for writing equations, it is not to be expected that high school pupils will master these symbols, except through well directed effort and hard work. Pupils in the elementary school spend much time during the first two or three years learning to write numbers in figures and in learning to read the numbers so written. It is necessary for them to do this. It is equally necessary for pupils in the high school to have much practice in interpreting the symbols used in algebra, and to extend this practice over a long period.

Perhaps this question arises: Are the x's, a's and b's used in algebra to be thought of as abbreviations for words, or do they stand for numbers? Before answering the question, it is necessary for us to get clearly before us an idea of what algebra, in its broader aspects, means. There is no limit to the number of possible algebras. Wherever a field of inquiry exists, a set of symbols or abbreviations may be devised, and a set of consistent principles may be determined to facilitate the investigation of questions arising in this field. That is, a certain type of algebra may be built up. For example, chemistry has its symbols. It also has its algebra. One writes the identity, $Mg + H_2O = MgO + H_2$, the symbols used being abbreviations for the names of certain chemical elements. This identity means that the "matter" referred to represents itself in two different forms, and such algebraic statements are of distinct aid in solving the problems of chemistry. Another type of algebra that may be of interest is that of George Boole, an English philoso-

pher and mathematician of about 1850. He invented an algebra that has the unique distinction of being the only algebra that has nothing to do with numbers. He denotes space by the symbol i. The letters a, b, c signify definite regions of space. $a + b$ means the portion of space made up of a and b, no portion counted twice. ab means that portion common to a and b. a^1 means space that is not a; thus $a + a^1 = i$. $a + a$, being the region made up of the regions a and a, of course, equals a. That is, $a + a = a$. Moreover, aa, being the region common to a and a, equals a. That is, $aa = a$. No coefficients or exponents are needed in Boole's algebra. Many of you are familiar with Hamilton's quaternions, a type of algebra in which the commutative law of multiplication does not hold. The hope has been expressed that algebras might be devised by means of which theological and political controversies might be settled through calculations with symbols, rather than through denunciations, discussions and debates. Then it would be proper for one politician to say to another, "Let us sit down and calculate." Now, the point which I am trying to make is that the x's, a's, and b's of algebra, considered in this general sense, do not stand for numbers. They stand for words and the words for which they stand may or may not refer to numbers.

In ordinary algebra the words referred to by the symbols are the names of things that have connected with them ideas of number. In this sense, then, the symbols of ordinary algebra refer to numbers; but they refer to numbers only in so far as the words, which they replace, stand for numbers. Much of the difficulty which the beginner has in understanding what algebra is "all about" may be obviated by emphasizing the notion that the symbols of algebra are abbreviations for words. $V = bh$ should mean volume equals base multiplied by height to the beginning pupil. It is hard for a pupil to see how b can stand for a particular number, without standing for this number or that one. It is easy to see that b stands for base, either, 17, 22, 25 or what not.

To sum up our discussion, let us say, that the second fundamental notion for a teacher of high school algebra to emphasize is that the symbols of algebra, highly developed as they are, have

as their purpose the clear, concise, usable expression of results of analysis, the symbols being abbreviations of the words used in expressing these results.

As was seen in the oft cited formula for the volume of a cylinder, the words and phrases were replaced by letters with a gain, both in the conciseness of the statement of the rule and in the clearness of its expression. Then, too, this formula, $V = bh$, may be made to yield other useful formulas, $b = \frac{V}{h}$ and $h = \frac{V}{b}$, if one merely applies to it the principle that, if the product of two factors and one of the factors are known, the second factor may be found by dividing the product by the known factor. Now the aim and end of all algebras, whether the algebra of chemistry, the algebra of political reformers, or the algebra of numbers is the same. It is to correct the weakness of language in expressing clearly and concisely the thoughts and ideas used in various fields of investigation, by expressing these thoughts and ideas in symbolic language. Yes, it is more than this. It is to so develop the symbols used, that they may be manipulated, conveniently, according to firmly established principles, to the end that they may be of aid in the investigation of questions arising in the different fields. It is just here, in my opinion, that the algebra used at present in the discussion of problems arising in educational tests and measurements is very deficient.

The symbols of ordinary algebra have a distinct advantage over the symbols of chemistry and of many other algebras in that they are manipulated according to the same principles as those used in arithmetic. As an example, let us consider the problem of squaring the mixed number, $3\frac{1}{2}$. The arithmetical work of solving this problem is carried out as follows: $3 \times 3 + 3 \times \frac{1}{2} + \frac{1}{2} \times 3 + \frac{1}{2} \times \frac{1}{2}$ and this may be written $3^2 + 2 \times 3 \times \frac{1}{2} + (\frac{1}{2})^2$. Analyzing the process of squaring $3\frac{1}{2}$, one sees that the form of the solution is quite independent of the particular number and fraction chosen for the problem. The multiplication is performed in the same way with any integral number and fraction. Stating the result of the analysis in words one has: The square of a number plus a fraction equals the square

of the number, plus the product of the number and the fraction, plus the product of the fraction and the number, plus the fraction squared. This is the same as: The sum of the square of the number, twice the product of the number and the fraction and the square of the fraction. If the words of the last two statements be abbreviated by the symbols of algebra, we have:

$$(n + f)^2 = nn + nf + nf + ff = n^2 + 2nf + f^2.$$

But now we notice that the work might have been much abbreviated, if we had dealt with the symbols, n and f, just as if they were figures. The symbols are of such a character that they correspond precisely to the figures used in the arithmetical calculation. It is not necessary to go through the process with the figures first, and afterwards to express the rule resulting from the analysis in symbols. The letters may be manipulated, as if they were figures, with just as much certainty of arriving at a true result.

The example given falls under the class of equalities called identities. An identity is essentially a declarative sentence. It is true, no matter what numbers are referred to by the symbols. In high school algebra and in mathematics beyond the high school much more use is made of identical transformations than the time spent in teaching the subject would indicate. It is an open question as to whether identities are not more important, both in high school algebra and in mathematics beyond the high school, than equations.

The principles used in writing identical expressions are simple and few in number: (1) Multiplying and dividing both numerator and denominator of a fraction by a number does not alter the value of the fraction. (2) Adding and subtracting the same number does not change the value of an expression. (3) Multiplying and dividing by the same number does not change the value of an expression. (4) Indicated processes may be performed, or processes to be performed may be indicated in an equivalent way, without changing the value of an expression. In teaching identities every step taken in the manipulation of the symbols should be proved by reference to some one of these principles. Then, perhaps, the error of subtracting the same

number from numerator and denominator of a fraction, and assuming that the value of the fraction remains unchanged, would not be so common among pupils.

I wish to discuss another example, under the general topic of the manipulation of symbols, somewhat different from the one already discussed. Let it be required that one find the divisor in a division problem, knowing that the dividend is 27, the quotient 4, and the remainder 3. The arithmetical reasoning involved is somewhat as follows: Since 4 times the divisor added to 3 gives 27, 4 times the divisor must be 3 less than 27, or 24. Then the divisor is 6. Writing the steps one has:

$$4 \times \text{the divisor} + 3 = 27$$
$$4 \times \text{the divisor} = 27 - 3, \text{ or } 24$$
$$\text{The divisor} = 6$$

That is, an equation has been solved completely.

The analysis of this particular equation indicates that whatever the dividend, quotient and remainder, the reasoning involved in finding the divisor is the same. If the essential features of the analysis be stated in words, and finally, expressed in symbols, one has:

$$qd + r = D$$
$$qd = D - r$$
$$d = \frac{D - r}{q}$$

Thus, in the case of the formula, one notices that the letters are used precisely as figures in arithmetic, according to the same principles, and with equal certainty of a true result. This, no doubt, is the reason for the feeling on the part of so many pupils that the symbols of algebra stand for numbers, rather than for words. Furthermore, it should be said that by manipulating the symbols of a formula according to the laws of arithmetic the mathematician may obtain results which lead to discoveries entirely unknown to the skilled electrician or mechanical engineer.

The equation differs from the identity in that it is, essentially, an interrogative sentence. Its members are equal for only certain values of the letters involved, and these are the numbers asked for by the equation. The process of finding these particular values is called solving the equation and the steps in the

solution are, as in the case of the identity, governed by principles in arithmetic. It is a lamentable fact that high school pupils do not know and understand the principles they use in solving equations, and in making identical transformations. They multiply such an expression as $\frac{x^2}{3} + \frac{2x}{3} - 5$ by 3 in order to simplify it, and assume that the resulting expression is equal to the first. They speak of the value of an equation, and say that multiplying each member of an equation by the same number does not change the value of the equation. Our pupils need to be clearer on the use of the following principles:

(1) If the same number be added to equals, the sums are equal.

(2) If the same number be subtracted from equals, the remainders are equal.

(3) If equals be multiplied by the same number, the products are equal.

(4) If equals be divided by equals, the quotients are equal.

(5) Multiplying or dividing the members of an equation by zero is not permissable.

Huntington in his article on "Fundamental Propositions of Algebra" says: "Until recently, high school algebra has been taught largely as a collection of rules for the manipulation of algebraic symbols. It has not at all been the developed science that elementary geometry has long since become. In fact, if it were not for the study of plane geometry in our high schools, it is doubtful whether or not our pupils would ever get from the study of algebra alone any clear notion as to what is meant by a mathematical demonstration, and yet algebra is better suited than geometry to show what is essentially involved in mathematical reasoning."

Now, having taken as the first fundamental element in algebra, the analysis of type problems in arithmetic, and as the second, a concise, usable symbolism for expressing the essentials of the analysis, we are ready to state a third fundamental notion that the teacher of high school algebra should emphasize. It is this: The symbols of algebra may be "exercised" like the figures in arithmetic and according to the same principles.

There is a fourth fundamental element which should be mentioned. The pupil in arithmetic, sooner or later, reaches the point, where, in his work in division, the dividend is not exactly divisible by the divisor. A new kind of number, called the fraction, is the result. Just so, in algebra, there comes a time when subtraction is no longer possible, and the necessity arises for a new kind of number, the negative number. Again, in arithmetic, numbers are found such that their square roots cannot be obtained, an absurd situation, and hence we have numbers called surds. On the other hand, in algebra, the pupil attempts to find the square root of -4, and comes in contact with another strange number called an imaginary.

Thus we have, as Huntington so well points out, not one science of algebra, but rather a collection of closely related sciences. We have the algebra of positive integers; the algebra of all integers, positive, negative, and zero; the algebra of positive rationals; the algebra of all rationals; the algebra of all real quantities, rational, irrational, positive, negative, and zero; and finally the algebra which includes all of the others, the algebra of complex quantities.

$4 \times 5 = 20$ is called multiplication. Also, $\frac{1}{2}$ of $10 = 5$ is called multiplication, although the pupil insists that, in the latter case, the number is divided. Again, in the problem of finding the compound amount of \$1.00 for 5 years at 4 per cent, one writes $(1.04)^5 \times \$1.00$. But, suppose that the time is $5\frac{1}{2}$ years. The amount at the end of $5\frac{1}{2}$ years cannot be found by $5\frac{1}{2}$ multiplications. It seems reasonable, however, that the process should be indicated by the same symbolism, namely $(1.04)^n \times \$1.00$, whatever this process may be.

The symbolism of algebra, originally invented to express the simple operations in arithmetic, is found so convenient, when numbers beyond positive integers are made necessary, that the definitions of these simple operations are deliberately made to include the less simple operations with the new kinds of numbers. This element of the gradual extention of the number concept and the need for a corresponding change in the definitions of algebraic operations is the fourth fundamental notion which I wish to point out.

I have tried to point out four fundamental elements which a teacher of high school algebra should have in mind:

(1) The ability to see the essential process in an arithmetical problem, that is, analysis.

(2) The clear, concise, usable expression in symbols of the rule resulting from the analysis.

(3) The manipulation of the symbols of ordinary algebra according to the principles of arithmetic.

(4) The extension of the number concept, and the need for a corresponding change in the definitions of fundamental operations.

In closing, I wish to acknowledge my indebtedness, for many of the ideas I have tried to make clear, to T. Percy Nunn, whose book on *The Teaching of Algebra* has been of great help to me.

RABBI BEN EZRA ON PERMUTATIONS AND COMBINATIONS

By JEKUTHIEL GINSBURG.

INTRODUCTORY NOTE: There are few algebraic topics more interesting to high school pupils that Permutations and Combinations when this subject is presented in an elementary fashion. It has of late years dropped out of the curriculum because it had come to be unnecessarily difficult through numerous unnecessary complications, but as a slight diversion it still has value, and in advanced algebra it is almost necessary.

Some interesting material relating to its presentation in early times has recently been found by Messrs. Ginsburg and Turetsky, and the editor of THE MATHEMATICS TEACHER has asked me to write a brief statement concerning it. In studying some unpublished manuscripts of Rabbi Ben Ezra (the learned Hebrew scholar of the 12th century, who is the subject of one of Browning's poems), Mr. Ginsburg found a curious motive leading to the study of combinations, namely, the desire of the astrologers to find the number of ways in which the planets could come into conjunction, this having an important bearing upon astrological predictions. The treatment is entirely distinct from any now in use, and it has been set forth in print only in the Hebrew language, and very imperfectly. Mr. Ginsburg has compared the text with an unpublished manuscript in the Hebrew Theological Seminary of New York, and has shown some very interesting results to which the work may easily lead. His translation and commentary are set forth in this article, and the material found by Mr. Turetsky will appear in a subsequent issue.

DAVID EUGENE SMITH.

The early history of the theory of permutations and combinations is one of the least explored fields of scientific research. The information on the subject given in the standard works on the history of mathematics is hardly sufficient to give even an approximate idea of the gradual development of this beautiful doctrine. Our knowledge about it is limited to a number of isolated facts chiefly relating to Christian Europe in the late Middle Ages and in modern times. Until recently the period immediately preceding the awakening of the interest in science in Europe has been a *tabula rasa* so far as this branch of mathematics is concerned. What did the Arabic scholars and those who came under their influence know about permutations and combinations? What utilitarian or ideal needs first attracted the attention of scholars to this subject? What influence, if any, did their work exert on later European developments? The information at our disposal does not allow us to answer any of these questions with the slightest degree of certainty. Most of

them will have to remain unanswered until further progress is made in the study of the rich literary treasures in the large European collections of Arabic, Hebrew, Persian and Hindu manuscript works of that period.

The object of the present article is to establish a few new points of departure for the further study of the subject, by calling attention to a number of hitherto unannounced facts the records of which have been preserved in the Hebrew literature, and especially to an extremely interesting mathematical fragment in a manuscript of the influential twelfth century astrologer and mathematician Rabbi Abraham ben Ezra (1093-1167), or, as he is known to the English-speaking world, Rabbi Ben Ezra.[1]

This fragment, which deals with the theory of permutations, is not found, as might have been anticipated, in one of his mathematical treatises, but in a work on astrology in which he discusses the influence of the stars on the destinies of the world.[2] This is interesting because it suggests that Rabbi Ben Ezra did not see any use for the theory of permutations outside of astrology. Most probably he was not aware, while solving a practical question of astrology by what we now call the theory of permutations, that he was writing upon mathematics and that his speculations had any theoretical value at all. This leads us directly to, and reveals in operation, one of the first causes that brought about the interest in combinations—the all-powerful influence of astrology. This should not surprise us. The apparent multiplicity of the powers of nature as represented by the stars acting severally and in conjunction with each other naturally led the astrologers to consider various combinations of stellar bodies and their possible influence upon human life. A similar mystic belief in the powers of the letters of the Holy

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[1] The form used by Browning in his famous poem.

[2] The name of the treatise is ha-Olam (the World). It is extant in manuscript form in a number of European libraries and also in the Jewish Theological Seminary in New York. In the New York copy the mathematical passage has been garbled by a somewhat ignorant scribe. Fortunately a portion of the work containing the passage referred to has been published in Hebrew from a manuscript in Berlin by D. Herzog in his edition (Tsophnath Pa-neakh) of Bonfil's supercommentary on Ben Ezra's work on the Bible (Heidelberg, 1911), with the view to elucidating an obscure passage in the supercommentary, but without pointing out its significance for the history of mathematics. In the following translation and discussion Herzog's printed version was used in conjunction with the New York manuscript copy.

Writ caused Jewish and Christian cabalists to develop a theory of permutations of their own, the details of which still await investigation. The work on permutations, written by one such cabalist, Moses Cordovero, has been the subject of careful study by Mr. Turetsky and will appear in a subsequent number of THE MATHEMATICS TEACHER.

The mathematical passage referred to is written in the vigorous and incisive style that characterizes the best of Ben Ezra's writings. The method of permutation used by him is original, and, as it seems to the writer of the present article, entirely unique.

Ben Ezra opens the discussion with a vigorous attack on the famous Arab astrologer Abu-Maschar,[1] who died in 886 A. D., about 100 years of age. "If thou hast found a book on conjunctions written by Abu-Maschar thou must not agree with him and thou shouldst not listen to him.[2] . . . Neither shalt thou trust, in the matter of conjunctions, to tables made by Hindu scholars, because they are altogether incorrect . . .[3] The right thing to do is to rely in each period upon tables made by contemporary scholars."

Ben Ezra then proceeds to discuss the number of possible conjunctions[4] between various members of the planetary system—a procedure that could not be very well undertaken without some scheme of combinations and permutations, and it is here that he develops his curious method.

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[1] Jafau ibn Mohammed ibn Omar al-Balkhi, Abu-Marschar. In the Latin Middle Ages he was known as Albumasar.

[2] His astronomical objections have been omitted in the translation as not bearing directly upon the subject of discussion.

[3] This testimony as to the state of Hindu science strikingly confirms the opinion held by a number of modern scholars who had less access to Hindu science than Ben Ezra. It also shows that the Hindu influence, which was very pronounced at the beginning of the Arabic period, faded out completely at the time of Ben Ezra.

[4] The astrologers believed the destinies of countries, nations, and individuals to be indicated in the heaven by the various positions of the planets. Two planets meeting at the same place in the heaven form a conjunction and exert a special significance on the development of events, the significance varying with the individual stars. A conjunction of three planets—that is, when three planets meet at the same place in heaven—has a greater influence than a conjunction of two planets, and so on. The most dreaded conjunction is of course the one of all seven planets referred to in an early Hindu tradition and expected to return in 26,000 years. A meeting of five planets was expected to cause great disturbances, inundations, plagues, etc., and the end of the world. For the year 1514 Stoffler predicted a terrible inundation on account of the conjunction of the superior planets. (See Hutton's Mathematical Dictionary.)

In view of the fact that this work by Ben Ezra has been published only in Hebrew and that no examination of its mathematical value has been made in any language, a complete translation and analysis is given below. In view of the peculiar style of Ben Ezra the explanations will be given after each closed paragraph.

"The number of conjunctions [of the seven planets] is 120. And this is how the number is found: It is known that the sum of the numbers from 1 to any desired number is found by multiplying it by half of itself and by $\frac{1}{2}$ of unity. For example if it is desired to find the number containing [all numbers] from 1 to 20, we multiply 20 by $\frac{1}{2}$ of itself which is ten [obtaining 200] and by $\frac{1}{2}$ of unity [*i. e.*, $20 \times \frac{1}{2} = 10$], and behold the result obtained is 210.[1]

"Now we shall proceed to find the number of binary conjunctions—that is, the combinations of two stars each. And it is known that there are seven planets. Now Jupiter has six conjunctions with the planets. Let us multiply then 6 by its half and by half of unity [$6 \times 3 = 18$, and $6 \times \frac{1}{2} = 3$]. The result is 21, and this is the number of the binary conjunctions."

It is characteristic of Ben Ezra's enigmatic way of writing that he should specify only the six conjunctions of Jupiter with the other planets, leaving for the reader to find out the source of the other members of the series

$$1 + 2 + 3 + 4 + 5 + 6 = 21$$

which he uses.[2]

"We wish now to know how many ternary combinations are possible. We begin by putting Saturn with Jupiter, and with them one of the others. The number of the others is five: mul-

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[1] In modern notation $1 + 2 + 3 + \ldots + n = n \,.\, \frac{1}{2}n + n \,.\, \frac{1}{2} = \frac{1}{2}n(n + 1)$.

[2] His reasoning was apparently this: Jupiter combines with each of the other six planets to form six conjunctions. Eliminating Jupiter the combinations of the remaining six planets with each other are now considered. Picking out one of them, for example Saturn, we find that, combining it with the other five planets, we obtain five combinations containing Saturn, and so on until the whole series $6 + 5 + 4 + 3 + 2 + 1$ is obtained.

tiply 5 by its half and by half of unity. The result is 15. And these are the conjunctions of Jupiter.[1]

"In the case of the conjunctions of Saturn[2] we have four planets left.[3] Multiply 4 by half of itself and by one-half. The result is 10.

"The conjunctions of Mars[4] are 3 multiplied by 2, amounting to 6.[5]

"The conjunctions of the sum[6] are 2 multiplied by $1\frac{1}{2}$ [that is by $\frac{1}{2}$ of 2 and by $\frac{1}{2}$], the result being 3,[7] and the con-

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[1] That is, Ben Ezra considered first the question as to how many of the ternary conjunctions contain Jupiter as a member. It is evidently the number of the conjunctions of the six other planets two at a time, since to each of these we may add Jupiter to form a ternary conjunction. But the number of these combinations (${}_6C_2$) is, according to the previous reasoning, $5 \cdot \frac{1}{2} \cdot 5 + 5 \cdot \frac{1}{2} = 5\left(\frac{5}{2} + \frac{1}{2}\right) = \frac{6 \cdot 5}{2} = 15$. The fact that Ben Ezra mentioned Jupiter and Saturn at the beginning suggests that his reasoning in proving the truth of this proposition was as follows: Taking first Jupiter and Saturn, we have five planets left (including, as was the custom, the sun and the moon—Uranus and Neptune being then unknown), which by combining successively with the combination Jupiter-Saturn will give five combinations. Taking now Jupiter and Mars, for example, leaving out Saturn, we have four planets left, which will, in combination with Jupiter and Mars, give four new ternary conjunctions. Taking now Jupiter and the sun, leaving out Mars and Saturn, the three remaining planets will give us three new combinations. Hence the number of ternary conjunctions, each of which contains Jupiter, is $= 5 + 4 + 3 + 2 + 1 = 5 \cdot \frac{1}{2} \cdot 5 + 5 \cdot \frac{1}{2} = 15$.

[2] That is, ternary conjunctions containing Saturn but not Jupiter.

[3] Using the reasoning developed in the case of conjunctions of Jupiter: After all the conjunctions containing Jupiter have been accounted for we have only six planets left. To find the number of combinations containing Saturn, we first fix Saturn and one of the other planets, e. g., Mars. By joining each one of the four remaining planets to the pair Jupiter-Mars, we get four conjunctions. Now joining Saturn with Venus, and leaving out Mars, we get three more combinations. Hence the number of conjunctions containing Saturn and not Jupiter is $1 + 2 + 3 + 4 = 4 \cdot \frac{1}{2} \cdot 4 + 4 \cdot \frac{1}{2} = 2\frac{1}{2} \cdot 4 = 10$.

[4] That is, $1 + 2 + 3 = 3 \cdot \frac{4}{2} = 3 \cdot 2 = 6$, the reasoning being similar to the above.

[5] That is, conjunctions containing Mars but not Jupiter and Saturn.

[6] That is, containing the sun but not any of the previously mentioned planets.

[7] That is, $1 + 2 = \frac{(1 + 2)2}{2} = 3$. It is interesting that even in a simple case like this Ben Ezra insisted on considering the work as involving a series.

junctions of the [three] lower planets[1] are one. All together [the number] is 35, and this is the number of the ternary combinations."

Writing down the series of numbers obtained by Ben Ezra in such a way as to record the arithmetical operations used, we quite unexpectedly obtain the following result:

$$ {}_7C_3 = \frac{1\cdot 2}{2} + \frac{2\cdot 3}{2} + \frac{3\cdot 4}{2} + \frac{4\cdot 5}{2} + \frac{5\cdot 6}{2} $$

whence we get the beautiful relationship

$$ 2\cdot {}_7C_3 = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6, $$

a relationship that could be easily extended to ${}_8C_2$, ${}_9C_2$, etc. In general,

$$ 2\cdot {}_mC_3 = 1\cdot 2 + 2\cdot 3 + \ldots + (m-2)\,(m-1), $$

a relationship not very helpful for permutations but which supplies an elegant method for the summation of the series

$$ 1\cdot 2 + 2\cdot 3 + \ldots + (m-1)m. $$

In the computation of the quaternary conjunctions Ben Ezra follows the same line of reasoning:

"We wish now to obtain the number of quaternary conjunctions. We shall begin with Jupiter, Saturn, and Mars. And since it is necessary to have three planets to join with it, the conjunctions begin with four.[2] Multiply by 2½ [*i. e.*, multiply 4 by ½ of 4 and by ½] and we obtain 10. Then follow the conjunction of Jupiter and Saturn with the others, and they will be multiplied by 2, amounting to 6; and behold it is now 16. Then Jupiter with Mars will begin, and there will be two [free planets] multiplied by 1½. Three is obtained. Then one more conjunction, hence the number of conjunctions of Jupiter is 20.[3]

Saturn begins with three.[4]

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[1] That is, conjunctions not containing the other four.

[2] That is, at first we get conjunctions by adding each of the four remaining planets to the group Jupiter-Saturn-Mars. Eliminating one of the three, say Mars, and putting in its place one of the four left, we shall have three planets left giving rise to three new conjunctions.

[3] The details of these computations are similar to those in the case of ternary conjunctions and can easily be followed by using the reasoning given in the footnotes to that case. The number of quaternary combinations containing Jupiter must evidently be equal to the number of ternary conjunctions of the remaining six planets, that is ${}_6C_3 = \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3} = 20$, which agrees with the result obtained by Ben Ezra.

[4] That is, excluding Jupiter, fixing three planets, say Saturn, Mars, and Venus, we have three possibilities of filling the fourth vacancy. Discarding one of the first three and replacing it by one of the unfixed three we get two choices, and so on, so that, in all, we have $3 + 2 + 1 = 6$.

Three times two is six.[1] Then 2 multiplied 1½ gives 3. Then one more conjunction. And then there are 10 conjunctions of Saturn.[2]

The number of conjunctions containing Mars but not Jupiter and Saturn is computed by Ben Ezra in the same way. "Mars begins with two non-fixed planets. Two times one and a half are three. Then one more conjunction. Together there are four quaternary conjunctions. Conjunction of the sun with the lower planets is one. All together, there are 35 quaternary conjunctions."

Ben Ezra's work, in computing the quaternary conjunctions, could be expressed by

$${}_7C_4 = {}_6C_3 + {}_5C_3 + {}_4C_3 + {}_3C_3 = 20 + 10 + 4 + 1$$

An interesting question arising out of this discussion is this: Was Rabbi Ben Ezra aware of the fact that the numbers 20, 10, 4, 1 are respectively the values of the members of the series

$${}_7C_4 = \frac{1\cdot 2\cdot 3}{6} + \frac{2\cdot 3\cdot 4}{6} + \frac{3\cdot 4\cdot 5}{6} + \frac{4\cdot 5\cdot 6}{6}$$

which is analogous to the two series

$${}_7C_2 = 1 + 2 + 3 + 4 + 5 + 6,$$

$${}_7C_3 = \frac{1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6}{1\cdot 2}$$

which he undoubtedly recognized as such? At the present stage of our knowledge we can only speculate upon the answer, although the brevity with which he treats the following cases suggests that he expected the reader to derive some sort of a rule from the previous discussion.

"We wish to find the quinary conjunctions. We find for Jupiter 15, for Saturn 5 and for Mars 1, together 21."[3]

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[1] $1 + 2 + 3 = 3 \cdot — = 3 \cdot 2 = 6.$

[2] That is, not counting Jupiter. The same number could be obtained by noting that the number of quaternary conjunctions containing Saturn, but not Jupiter, is ${}_5C_3 = \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} = 10.$

[3] In modern terms: the number of conjunctions of planets containing Jupiter as a member is ${}_6C_4$, or 15. Those containing Saturn but not Jupiter are ${}_5C_4$, or 5. Those containing Mars but not the previous two are ${}_4C_4$, or 1.

Here again the numbers given by Ben Ezra fit into the identity

$$_7C_5 = \frac{1\cdot 2\cdot 3\cdot 4}{4!} + \frac{2\cdot 3\cdot 4\cdot 5}{4!} + \frac{3\cdot 4\cdot 5\cdot 6}{4!},$$

which leads to the relationship

$$4!\,_7C_5 = 1\cdot 2\cdot 3\cdot 4 + 2\cdot 3\cdot 4\cdot 5 + 3\cdot 4\cdot 5\cdot 6.$$

The remaining groups of conjunctions are treated not quite as fully as the previous ones. "The senary conjunctions are six for Jupiter and one for Saturn,[1] and there is one conjunction of all the seven planets."

This could be expressed in modern notation as follows:

$$_7C_6 = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{5!} + \frac{2\cdot 3\cdot 4\cdot 5\cdot 6}{5!},$$

$$_7C_7 = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{6!}.$$

Rabbi Ben Ezra concludes with the following remark: "The total number is 120 conjunctions; all component [groups of] conjunctions are odd in number and they are divisible by 7."

A generalization of the above results leads to the following interesting relations:

$$_mC_2 = 1 + 2 + 3 + \ldots + (m-1) \quad (1)$$

$$2!\,_mC_3 = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + (m-2)(m-1) \quad (2)$$

$$3!\,_mC_4 = 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + (m-3)(m-2)(m-1) \quad (3)$$

$$4!\,_mC_5 = 1\cdot 2\cdot 3\cdot 4 + 2\cdot 3\cdot 4\cdot 5 + \ldots + (m-4)(m-3)(m-2)(m-1) \quad (4)$$

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$$r!\,_mC_{r+1} = 1\cdot 2\cdot 3 \ldots r + 2\cdot 3 \ldots (r+1) + \ldots + (m-r+1)(m-r+2) \ldots (m-1) \ldots \quad (5)$$

The first two were clearly recognized as series by Ben Ezra, and were treated by him as such; the others are suggested by the method he employed. Crude as these relations are for the subject itself, they are very helpful in the summation of series of products of natural numbers. In fact there is hardly another method as convenient and as easy to remember as the one ob-

[1] That is, counting Saturn but not Jupiter.

tained by substituting $m + 1$ for m in (1), $m + 2$ for m in (2), $m + 3$ for m in (3), and so on. In this way we obtain the following set of formulas:

$$1 + 2 + 3 + \dots + m = {}_{m+1}C_2$$
$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + m(m + 1) = 2!\, {}_{m+2}C_3$$
$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots + m(m + 1)(m + 2) = 3!\, {}_{m+3}C_4$$

From this we can easily deduce the following general theorem: *The sum of the first m terms of the series of products of natural numbers taken in order of magnitude r at a time is equal to $r!\, {}_{m+r}C_{r+1}$.*

The above set of formulas allows us also to find the sum of the powers of the series of natural numbers in a more elegant way than is ordinarily used. We notice that (5) may be written

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + m(m + 1) &= 1(1 + 1) + 2(2 + 1) + 3(3 + 1) + m(m + 1) \\ &= (1^2 + 2^2 + 3^2 + \dots + m^2) + (1 + 2 + 3 + \dots m) \\ &= 2!\, {}_{m+2}C_3. \end{aligned}$$

$$\begin{aligned} \text{Hence } 1^2 + 2^2 + 3^2 + \dots + m^2 &= 2!\, {}_{m+2}C_3 - (1 + 2 + 3 + \dots + m) \\ &= 2!\, {}_{m+2}C_3 - {}_{m+1}C_2. \end{aligned}$$

For the sum of the cubes consider the identity (6)

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots + m(m + 1)(m + 2) = 3!\, {}_{m+3}C_4$$

The general term being of the form $a(a + 1)(a + 2) = a^3 + 3a^2 + a$, we may write it

$$(1^3 + 2^3 + \dots + m^3) + 3(1^2 + 2^2 + \dots + m^2) + 2(1 + 2 + \dots + m) = 3!\, {}_{m+3}C_4.$$

Substituting for the sum of the squares and first powers the values previously obtained, and then transposing, we have

$$1^3 + 2^3 + \dots + m^3 = 3!\, {}_{m+3}C_4 - 3 \cdot 2!\, {}_{m+2}C_3 + {}_{m+1}C_2$$

In a similar way we find that

$$1^4 + 2^4 + \dots + m^4 = 4!\, {}_{m+4}C_5 - 6 \cdot 3!\, {}_{m+3}C_4 + 7 \cdot 2!\, {}_{m+2}C_3 - {}_{m+1}C_2,$$
$$1^5 + 2^5 + \dots + m^5 = 5!\, {}_{m+5}C_6 - 10 \cdot 4!\, {}_{m+4}C_5 + 25 \cdot 3!\, {}_{m+3}C_4 - 15 \cdot 2!\, {}_{m+2}C_3 + {}_{m+1}C_2,$$
$$1^6 + 2^6 + \dots + m^6 = 6!\, {}_{m+6}C_7 - 15 \cdot 5!\, {}_{m+5}C_6 + 65 \cdot 4!\, {}_{m+4}C_5 - 90 \cdot 3!\, {}_{m+3}C_4 + 31 \cdot 2!\, {}_{m+2}C_3 - {}_{m+1}C_2.$$

The above examples will give the reader a general idea of the power of the method discovered by Ben Ezra but used by him

in only a single case, and that case the very one in which it is least effective. This method was soon abandoned and, in the fourteenth century, we find Levi ben Gerson developing a theory of combinations very much like the one in present use.[1]

The results suggested by the fragment discussed above are as follows:

1. In the twelfth century there already existed a crude theory of combinations which owed its existence chiefly to the influence of mysticism.

2. The method used by Ben Ezra in finding the number of combinations of m elements taken n at a time was by reducing it to combinations of lower order according to a rule which may now be expressed as follows:

$$ {}_mC_n = {}_{m-1}C_{n-1} + {}_{m-2}C_{n-1} + {}_{m-3}C_{n-1} + \cdots + {}_{n-1}C_{n-1} $$

3. Ben Ezra was aware that the number of combinations of m elements taken 2 at a time is equivalent to the sum of the series

$$ 1 + 2 + \cdots + (m - 1), $$

and that

$$ {}_mC_3 = \frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2} + \cdots + \frac{(m-2)\ (m-1)}{2} $$

but whether he knew the rule for higher cases is an open question.

4. Ben Ezra's work on permutations was one step in the history of the development of the theory of combinations, a process that was perfected in the time of Gersonides and his immediate successors.

[1] Attention to his work on permutations has been called by G. Eneström, Bibl. Math., Vol. XIV (3), p. 276, but a thorough discussion of his achievement on this line is still lacking.

DISCUSSION

Introducing Mechalus to Geometry. In ancient times the few greatest scholars of a generation studied geometry; it was a senior study at Yale until 1742; but now we expect Mechalus Pezanoski, aged 16, who came over in the steerage from Poland when he was a baby, to grasp demonstrative geometry in one year. Obviously, our task of teaching that geometry is vastly different from that of our predecessors.

Mechalus was sent to school because the state law requires his presence there until he is 17. He has no desire to study geometry in order that he may learn formal reasoning; he lives in a concrete world dominated by the movies and his transition to the rarified atmosphere of abstract thought must be gradual or his brain will become temporarily paralyzed. Happily compasses, a protractor, and ruler are concrete things and if they are used to construct the same geometric figures he met in elementary mathematics, his introduction to the new subject is safely begun.

The next difficulty he will encounter is the group of necessary preliminary axioms and definitions. Several axioms he has already met in algebra; properly encouraged he may formulate others from them, and vivid illustrations will fix the most difficult in his mind. For instance, if John, the tall athlete, medium-sized Harry and Fred, the class pigmy, are lined up in a row and renamed "things," Mechalus will see very clearly that if the first of three things is greater than the second, and the second is greater than the third, the first is greater than the third.

When after many intuitional exercises the formal demonstration of a theorem actually begins, Mechalus is likely to look at his teacher with a feeling akin to pity when she proves with much labor that two certain triangles are congruent when he knew they were congruent by merely looking at them. But he may be induced to feel there is some need for proof if several eye-deceiving diagrams are drawn which appear to have one relation and may be easily proved to have another. He then may be quite willing to cut out paper triangles whose specified measurements show the parts which are given equal, and use

them with his newly acquired rules for the new game to prove that these triangles may be made to fit and are therefore congruent.

He next encounters his old enemy English when he finds that he must write out this proof. But here again his teacher proves to be an ally and gives him the following skeleton outline which he uses for every proof:

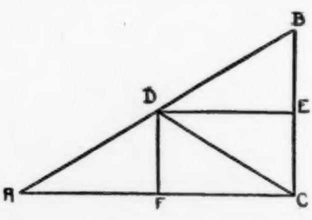
1. Statement of theorem to be proved.
2. Figure.
3. Given.
4. To prove.
5. Proof:

   Arguments Reasons.
6. Statement of theorem proved.

Together they work out the complete written proof and Mechalus copies it in his note book for a model.

Much the same method of procedure is followed for several days; gradually Mechalus finds it unnecessary to cut out paper triangles, he can see them plainly enough in the drawings he makes; gradually the teacher lessens the amount of proof in the models put on the blackboard and Mechalus completes them in his notebook. In order that this notebook may be a trustworthy guide he hands it in each week for correction.

Mechalus enjoys the many easy exercises which are given beginning with the first theorem, and later in his course feels like a creator of mathematics if he can discover for himself theorems which are found on later pages of the textbook. To this end the teacher may make an occasional assignment like this:



Given the right triangle ABC with CD connecting D the midpoint of AB with C, DE parallel to AC, and DF parallel to BC.

State all the theorems that appear to be true and try to prove them.

If geometry continues to compete successfully with the movies the recitations must show variety. Historical bits may be used to throw glamour on cut and dried theorems and Mechalus is entranced with the report of the Bright Boy who tells about Pythagoras and his mystic fraternity. On a Monday morning he may be awakened by a few minutes spent on foolish thought problems like the following: If 2 horses can jump a ditch 3 feet wide, how wide a ditch can 8 horses jump? Another good alarm clock is the apparently correct proof of the obviously impossible facts that every triangle is isosceles and that an obtuse angle equals a right angle.

Algebra is the first aid for many theorems. For example, in the problem, "To construct a square equal in area to a given triangle," if Mechalus makes use of his old friend x for the unknown side of the square, a and b for the altitude and base of the triangle, he can with little difficulty translate his problem into the equation x^2 equals $\frac{1}{2}ab$. This easily suggests the proportion $\frac{1}{2}a/x$ equals x/b, which, translated into geometry states that the side of the square may be found by constructing the mean proportional between $\frac{1}{2}a$ and b.

But Mechalus is as lazy as he dares to be; he is quite willing to take a chance that the Bright Boy will be called upon to recite the more difficult theorems. In order to make him feel responsible for each of the fundamental theorems the following scheme was devised and worked quite successfully:

At the beginning of each small unit of work the most important theorems were assigned to be recited within a specified time by each member of the class. After the pupil had mastered the theorem from the meagre suggestions in the text, he put the figure on the board and wrote his name on a slip of paper on the teacher's desk to show he was ready to recite. The recitation was given to the teacher or to one of her assistants, the Bright Boys, who had already given a perfect demonstration of the proposition.

The advantages of the plan were several. Every pupil knew that he must recite every assigned theorem, and therefore attempted to prepare it; every pupil had the ambition to be an

assistant teacher so he aimed for a perfect instead of a passing demonstration; every pupil could progress at his own pace, and ample provision could be made for additional work for the Bright Boy; make-up work for absences could be more easily checked up; initiative was developed and enough interest was aroused so that the closing bell was greeted with groans instead of cheers.

The last half or two-thirds of the class time was usually devoted to this sort of free-for-all recitation; the first part was spent in the conventional manner of anticipating difficulties, correcting recurring errors, reciting and discussing originals and theorems which comprised the additional assignment for that day's work.

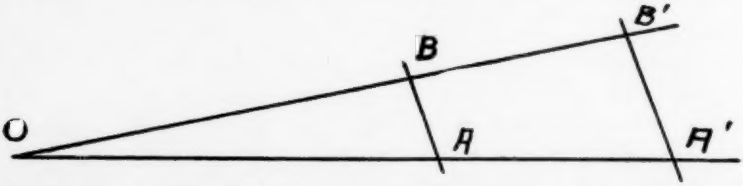
By this time Mechalus has become sufficiently interested in geometry and accustomed to its abstract ideas and formal reasoning so that he bears the necessary if unpleasant reviews, outlines, drills, and even examinations with fortitude if not with pleasure.

MARY A. POTTER.

Racine, Wisconsin.

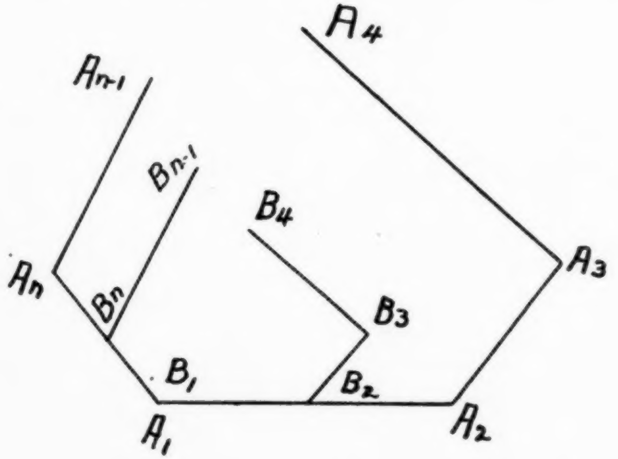
Notes on Mr. Evans' Paper in the March Teacher. One cannot help but feel grateful to Mr. Evans and the TEACHER for a paper which does not hesitate to break with tradition. We are the more grateful because of the recent command to fall down and worship tradition—or international authority, as it is now called.

In a paper having such a large sweep it is to be expected that there will be matters which will not appear to all to hang well



together. One such discrepancy is Mr. Evans' point of view concerning what he calls "perspective" position and the early stages of his plan. He says, "No change in the definition of similar triangles; leave that as it is." ***But is it not a fact that in order to prove the theorems on similar triangles this same***

perspective position must be employed. And if the bond *must* be formed *in the beginning* between the definition of similar triangles and the perspective position is it not better psychology to continue this bond than to break it. The writer maintains that the figure below is essentially in perspective position.



Furthermore, in two dimensions it may be doubted whether or not there is any real need ever to break the bond between similar polygons and perspective position. Mr. Evans condemns this bond as unnatural and inadequate. His words are, "Against the general use of such a definition and method, however, is the fact that similarity belongs to the figures themselves, and not to their relative position." As to *method*, and with the *present definition*, it seems that Mr. Evans' position may be questioned on the basis of fact. As a method the perspective position seems entirely adequate, as in the last figure.

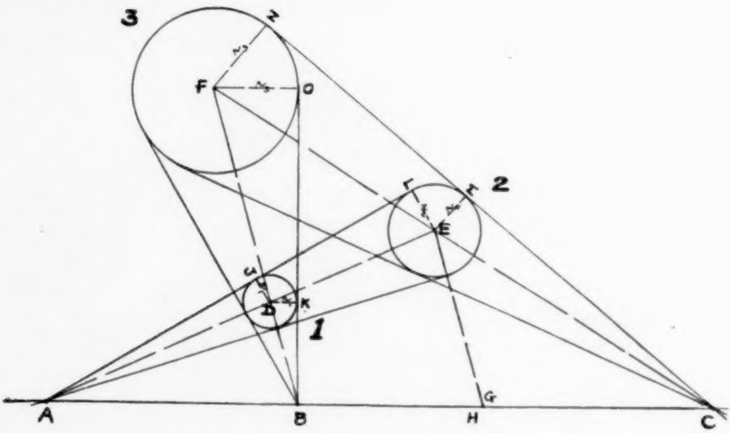
Montclair, N. J. H. F. HART.

Original Solution in Plane Geometry.

Problem: The points of intersection of external tangents drawn between any two of three circles of different sizes, in turn, lie on a straight line passing through these points.

Given the three circles No. 1, No. 2 and No. 3; and the external tangents between the circles No. 1 and No. 2, intersecting at the

point A; those between circles No. 1 and No. 3 intersecting at the point B; those between the circles No. 2 and No. 3 intersecting at C.



To Prove: That the points A, B and C lie on a straight line ABC passing through the points A, B and C.

Proof: Let r_1, r_2 and r_3 be the radii of the circles No. 1, No. 2 and No. 3, respectively, and D, E and F their centers.

In the similar triangles ADJ and AEL,

(1) $$\frac{AD}{AE} = \frac{r_1}{r_2}$$

In the similar triangles BDK and BFO

(2) $$\frac{BD}{BF} = \frac{r_1}{r_3}$$

In the similar triangles CEM and CFN

(3) $$\frac{CE}{CF} = \frac{r_2}{r_3}$$

Combining equations (1) and (3) by multiplication, we have

(4) $$\frac{AD}{AE} \times \frac{CE}{CF} = \frac{r_1}{r_2} \times \frac{r_2}{r_3} = \frac{r_1}{r_3}$$

Equating equations (2) and (4), we have

$$\frac{BD}{BF} = \frac{AD}{AE} \times \frac{CE}{CF} \text{ or}$$

(5) $$\frac{DB \times AE}{AD} = \frac{CE \times FB}{CF}$$

From the point E, draw the line EG parallel to the line FB and meet the straight line AB produced at the point G.

Then in the similar triangles ADB and AEG

(6) $$\frac{AD}{AE} = \frac{DB}{EG}$$

Connect the points B and C by a straight line. Draw the line EH parallel to FB and to meet the straight line BC at the point H.

Then in the similar triangles CEH and CFB.

(7) $$\frac{CE}{CF} = \frac{EH}{FB}$$

From equation (6)

(8) $$EG = \frac{DB \times AE}{AD}$$

From equation (7)

$EH = \dfrac{CE \times FB}{CF}$, but from (5) $\dfrac{DB \times AE}{AD} \times \dfrac{CE \times FB}{CF}$.

Therefore, $EG = EH$.

But EG and EH are equal and parallel; therefore EG and EH coincide and the points G and H are one and the same point.

Hence, the straight line BG (BH) is common to both the straight line AB produced and the straight line BC.

Hence, the straight line BC is the continuation of the straight line AB.

Hence, the points A, B and C lie on the same straight line ABC.

Q. E. D.

CAPTAIN ROBERT A. LAIRD.

Corps of Engineers, United States Army.

An extract from the autobiography of Herbert Spenser, page 187, volume 1, adds interest to this proof: "It may be remembered that ah early chapter states that when seventeen, I hit

on a geometrical theorem of some interest. This remained with me in the form of an empirical truth; but during the latter part of my residence in Worcester, responding to a spur from my father, I made a demonstration of it, and now, that it had developed this form, it was published in the *Civil Engineer and Architects Journal* for July, 1840. It is produced in Appendix B. I did not know at the time that this theorem belongs to that division of mathematics at one time included under the name of Descriptive Geometry, but known in more recent days as the geometry of position—a division which includes many marvellous truths. Perhaps the most familiar of these is the truth that if to three unequal circles anywhere placed, three pairs of tangents be drawn, the points of intersection of the tangents fall in the same straight line—a truth which I never contemplate without being struck by its beauty at the same time that it excites feelings of wonder and of awe; the fact that apparently unrelated circles should in every case be held together by the plexus of relations, seeming so utterly incomprehensible. The property of a circle which is enumerated in my own theorem, has nothing like so marvellous an aspect but is nevertheless sufficiently remarkable."

Professor Hedrick's Report on the Function Concept in Elementary Mathematics. Of the various preliminary reports which have been issued by the National Committee on Mathematical Requirements, none is more provocative of thought than the report on the function concept. This was to be expected by those who have the good fortune of personal acquaintance with the author. In direct ratio to his high standing among mathematicians in this country, and his familiarity with the various problems in the field of secondary instruction, is the degree of boldness exhibited by one who ventures to disagree with him at certain points.

Or rather, not so much to disagree with his conclusions as to question the practicability of many of his recommendations, however desirable they may be from the point of view of an inclusive study of the whole problem. In other words, it is the desire of the writer to ascertain just what Professor Hedrick is "letting us in for."

He takes as his text a quotation from the first report issued by the committee, as follows "*The one great idea* which is sufficient in scope to unify the course is that of the functional relation."

Let it be said at once that this is the one idea which at the present time in secondary mathematics is most likely to be thrown out of the window.

Is there any one idea about which the present course in elementary algebra (one or two years), is centered? I think that there is, and that it is the idea of *transformation.* Is it not true that the so-called algebraic *operations* are really transformations in accordance with laws which define properties? The whole emphasis in the past has been upon the universal character of these laws.

(Or rather, it should have been so. In too many instances the laws have been lost sight of in a welter of technique, with examples and methods based for the most part upon imitation.)

Elementary algebra has in the past been almost wholly static. Efforts to render it otherwise have been feeble and generally futile, for the reason that to do so successfully involves the upsetting of the whole program. Professor Hedrick shows clearly enough what is the elementary approach to the subject. The writer's point is that it is doubtful if occasional examples would serve any very useful purpose unless the whole subject were recast along the lines recommended.

If this were done, we would have a subject much more interesting than our present possession. Professor Hedrick indicates that it is not necessary to *wait* for the study of graphs before introducing the function idea. This is true, but in a course which has been unified about the functional relation the graph would be the *first* topic and others would cluster around it. Negative numbers, irrational numbers, and various algebraic forms would then be interpreted in their relation to the continuity of the graph. This development is, in the opinion of the writer, entirely within the compass of the secondary education from its beginning. But the usual sequence of topics would be sadly shaken. And unless the time limits were greatly extended, much of what is now regarded as essential to a second-

ary course, in the way of extrication of the "unknown" from an intricate mass of symbolism, must of necessity go by the board.

To illustrate—ratio and proportion and the progressions would be among the first topics considered in the ninth year course, with fractional and negative exponents an immediate sequence of the negative number. The polynomial, its factors, and operations upon it would appear much later and be stressed very lightly. Logarithms would come early in the course, and algebraic fractions near the end. Functions which in general are defined by algebraic fractions are rather difficult to handle.

The graph at present occupies a somewhat anomalous position in the secondary course. It is usually presented as the picture, not of a function but of an equation in two variables in which the functionality of both variables is implicit. For this reason the relation of the graph to the algebraic form appears to the student as more or less accidental.

If the function (and presumably its graph) is to be the one great idea which unifies the course, it should be developed much further in secondary mathematics than is done at present, or is suggested by Professor Hedrick. After all, the *mathematics* of the function idea is concerned with the definition of the function. To fail to do this, so to speak, takes all of the fun out of the function! It is all very well to say, for example, that horse-power is a function of "R.P.M.," but the question is "What function?" and the topic loses interest unless the answer is forthcoming. In the case of the steam engine it is a linear function. In the case of a six-cylinder automobile engine it is something quite different, to say the least. Progress in the subject would be represented in a measure by the ability to define functions, and this would lead to such topics as recurring series, and the method of finite differences, presumably in the fourth high school year, and to the early consideration of Fourier series and other series in the college course. The writer would state in passing that he has taught the first two topics mentioned in a fourth year high school course to the great satisfaction of the students, and to his own, but without any alarming degree of logical precision!

And this brings us to the crux of the whole question. In the past it has been incumbent upon the secondary teacher (ostensibly) to set for his students a standard of logical precision. It was this thing and nothing else that gave Euclid such supreme importance in the eyes of the typical English schoolmaster. It is this which has led eminent lawyers, physicians, and clergymen to descant upon the glories of plane geometry in early schooling, in spite of the fact that in most American texts Euclid has been so terribly mishandled that the vaunted logical precision of elementary geometry is a philosophical shriek.

And the spirit of Euclid has dominated the teaching of algebra in America in spite of greater abuse in point of precision. Up to fifteen years ago, so the best authority informs us, the study of Euclid preceded that of algebra in English schools, for the most part. In America the order of the subject was reversed, but the aims and ideals were retained, again for the most part. The whole spirit of Felix Klein's proposed reforms in secondary mathematical teaching looked away from the hard-boiled formalism of an earlier generation, and toward a broadening of the vision of the significance of mathematics in the works of nature and in the field of human endeavor. That Klein himself recognized the dangers lurking in any procedure in this direction is evidenced by a conversation with him, reported by Professor D. E. Smith, touching upon this very point. The vital point remains, nevertheless. If the secondary course is to be centered about the function concept, considerations of logical precision will of necessity be in no small measure postponed to advanced courses in college and graduate school.

And this brings us to geometry, where the writer must take definite leave of Professor Hedrick. He apparently thinks that it is practicable to mention functions incidentally to the customary development of plane synthetic geometry. This would only make the usual geometric confusion worse confounded. We have dropped completely (in America) Euclid's theory of proportion and measurement, a theory which, as Dean Fine has pointed out, is perhaps his most brilliant achievement, foreshadowing the recent researches of Kronecker. But we have retained, out of respect for the classical tradition, the previous

development necessitated by this theory, and have worried ourselves half sick over the axiom of parallels. If geometry is to share in the unification around the function concept, it means at the very outset the establishment of a correspondence between points and number forms, and that means the basing of the subject upon Cartesian analysis. An *elementary* analytic geometry is no more difficult than an *elementary* synthetic geometry. The whole question turns upon what is elementary, and that is a matter to be decided by experience. Geometry then assumes its proper place in the mathematical sequence. Probably the experiment has never been tried. But is it not likely that the general idea of a set of transformations is reducible to elementary terms? In this way the function concept would dominate the subject, and motion would assume its rightful place in the geometric scheme, instead of being, as it is now, a sheet anchor, under the alias of "superposition," to save from logical shipwreck, in spite of the fact that in a geometry which is wholly static, as ours sets out to be, motion has no place at all.

In such a development, Klein's definition of a geometry as a set of properties which remain invariant under a given transformation takes on an elementary significance. We would have a geometry of parallelism, of symmetry, of point reflection, of areas, of similarity, and possibly some others. The preparation of a text of this character would be an interesting experiment. Judging by past performances, however, it will be some time before such a text has any vogue in secondary circles. In the meantime, the function concept may as well be left out of a secondary course in geometry.

As far as trigonometry is concerned, the function concept lies at the basis of the subject, and there is no ancient canonical sequence to hinder its free development. If, in actual classroom practice, the trigonometric functions are treated merely as the properties of fixed angles, it is an evidence of imperfect acquaintance with the subject on the part of the instructor or the textbook writer. There are many evidences of this in certain instances. The report does well to remind teachers what a trigonometric function really is.

Harrison E. Webb.

Central High School, Newark, N. J.

NEWS AND NOTES

PRESIDENT JOHN H. MINNICK and the Executive Committee of the National Council of Teachers of Mathematics have perfected plans to present the interests of the Council to practically all organizations of mathematics teachers in the United States. State Representatives have been appointed, as follows:

Alabama—C. G. Bandman, Central High School, Birmingham
Arizona—A. L. Hartman, Mesa Union High School, Mesa
Arkansas—George W. Drake, University of Arkansas, Fayetteville
California—Gertrude E. Allen, University High School, Oakland
Colorado—E. L. Brown, Northside High School, Denver
Delaware—Mrs. Elinor B. Rosa, Milford
District of Columbia—Harry English, Board of Examiners, Washington
Florida—Miss Olga Larson, Box 84, Apopka
Georgia—George W. Brindle, Surrency
Idaho—Winona M. Perry, 719 Sherman Ave., Couer D'Alene
Illinois—R. L. Modesitt, 1703 S 7th St., Charleston
Indiana—Walter G. Gingery, Shortridge High School, Indianapolis
Iowa—Ira S. Condit, Iowa State Teachers College, Cedar Falls
Kansas—Miss Inez Morris, 728 State St., Emporia
Kentucky—V. D. Roberts, Somerset
Louisiana—Jeanne Vautrain, 1820 N. Rampan St., New Orleans
Maine—E. L. Moulton, Edward Little High School, Auburn
Massachusetts—William H. Brown, High School, Amherst
Michigan—John P. Everett, Western State Normal School, Kalamazoo
Minnesota—W. D. Reeve, 828 University Ave., Minneapolis
Mississippi—Miss Clyde Lindsey, Oxford
Missouri—Charles Ammerman, McKinley High School, St. Louis
Nevada—Miss Bertha C. Knemyer, Elko Co. High School, Elko
New Mexico—T. C. Rogers, 1018 Fourth St., E. Las Vegas
New York—Raleigh Schorling, 423 West 123rd St., New York City
Ohio—Miss Florence M. Brooks, Fairmount Jr. High School, Cleveland
Oklahoma—C. E. Herring, Box 489, Oklahoma City
Oregon—Florence P. Young, Franklin H. S., Portland
Rhode Island—P. R. Crosby, 110 N. Bend St., Pawtucket
South Carolina—J. Bruce Coleman, University of South Carolina, Columbia
South Dakota—Iona J. Rehn, 735 S. Summit Ave., Sioux Falls
Tennessee—F. L. Wrenn, McCallie School, Chattanooga
Texas—J. O. Mahoney, 1900 Crockett St., Dallas
Vermont—Llewellyn R. Perkins, 6 Franklin St., Middlebury
West Virginia—Miss Blanche Stonestreet, 591 Spruce St., Morgantown
Wisconsin—Miss Mary A. Potter, Racine High School, Racine

These representatives are actively engaged in urging the teachers in their respective states to affiliate with the Council, and to participate, in a more direct way, in the reorganization movement now being effected in mathematical education. A special circular has been prepared to set forth the purposes and values of the Council. Copies may be secured from your representative, from Mr. John A. Foberg, Secretary-Treasurer, Camp Hill, Pa., or from President John H. Minnick, School of Education, University of Pennsylvania, Philadelphia.

THE Mathematics Section of the Indiana State Teachers' Association has prepared the following program for its annual meeting to be held at Indianapolis, October 19, 1922:

I. Mathematics for Discipline and Knowledge: Professor E. N. Johnson, Butler College, and R. R. Cromwell, Anderson High School.

II. High School Mathematics Clubs: Professor Cora B. Hennel, Indiana University, and Mr. W. H. Carnahan, Washington High School.

III. The Reorganization of Secondary School Mathematics: Professor W. D. Reeve, University of Minnesota.

THE twenty-second meeting of the Central Association of Science and Mathematics Teachers will be held at the Hyde Park High School, Chicago, Illinois, December 1 and 2, 1922. A program unusually strong in the prominence of its speakers, and in the wide range of interests covered, is offered for both the general and sectional meetings. Professor Theodore Soares, of the University of Chicago, noted for his appealing eloquence, will speak on the Social Values in the School Curriculum. Dean M. E. Haggerty, of the University of Minnesota, one of the foremost authorities in the country on educational measurements, will discuss the Place of Measurement in the Solution of Educational Problems in High School Science and Mathematics. Professor Otis W. Caldwell, the noted head of the Lincoln School, New York City, will deliver three addresses. Professor John M. Coulter, of the University of Chicago, is so widely known and appreciated that every one will want to hear him on Changing Ideals in Science Teaching. Teachers from the high school class room will discuss methods and progress in their own work before the various sections.

The social interests of the members and their guests have not been overlooked. Abundant opportunity will be given for gaining information and inspiration from friendly personal conferences with teachers from various parts of the country.

This is the twentieth anniversary of the organization of the Association. Come to the meeting with a determination to make it a conspicuous milestone in the history of the organization. Help us to lead in educational progress as we have done in a conspicuous way during the past two decades.

Every teacher of science and mathematics owes it to himself and to his fellows to support teachers organizations. Help make this Association what you think it ought to be. If you are a member get a new member.

The railroads are offering special rates. Watch for the Year Book. Make your plans early to attend the meeting.

Soldan High School, ALFRED DAVIS,
St. Louis, Mo. *President.*

THE seventh summer meeting of the Mathematical Association of America was held at the University of Rochester, Rochester, New York, on Wednesday and Thursday morning, September 6-7, 1922.

The program which was announced by the program committee follows:

WEDNESDAY

10:00 A. M. Present Status of Unified Mathematics.

1. "The Problem of Organizing Freshman College Courses"—Professor J. W. Young, Dartmouth College.

2. "Historical Consideration of Unified Mathematics,"—Professor L. C. Karpinski, University of Michigan.

3. "Some Aspects of Unified Mathematics for Freshmen,"—Professor R. W. Burgess, Brown University.

4. "Internal Reasons for Unification,"—Professor C. E. Comstock, Bradley Polytechnic Institute.

5. General dicussion, led by Professor K. D. Swartzel, University of Pittsburgh, and Professor C. H. Yeaton, Oberlin College.

2:00 P. M. 6. Presidental Retiring Address: "Contradictions in The Literature of Group Theory,"—Professor G. A. Miller, University of Illinois.

7. "Mathematical Puzzles as an Introduction to Investigation,"—Professor W. B. Carver, Cornell University.

8. "An English Text on Mathematics written about 1810," —Professor Elizabeth B. Cowley, Vassar College.

9. "Impressions of Mathematics and Mathematical Instruction in Italian Universities,"—Professor Virgil Snyder, Cornell University, by invitation.

10. "The Present Status of the Formal Discipline Controversy,"—Professor N. J. Lennes, University of Montana.

THURSDAY.

9:30 A. M. Meeting at the Research Laboratory of the Eastman Kodak Company, as guests of the Company.

1. "The Application of Vectors to Problems of Geometrical Optics,"—Dr. Ludwig Silberstein.

2. "The Physical Problems Involved in Photographic Research,"—Mr. L. A. Jones.

3. "The Calculus of Probability and Theory of Light Quanta Applied to the Problem of the Latten Image," Dr. Silberstein.

4. Inspection of the Research Laboratory.

THE teachers of Mathematics in southern Massachusetts hold a number of Saturday morning meetings during the school year. Among the topics discussed last year were:

1. How often should tests be given? Miss Louise Bullard, Taunton.

2. Should tests be too long for pupils of average ability to finish? Miss E. Estelle Miles, Fall River.

3. Should there be tests where absolute perfection is required on a question? Miss Mary F. Hitch, New Bedford.

4. Topics which will broaden the outlook of teachers of mathematics in secondary schools, Professor R. D. G. Richardson, Brown University.

5. The Mannheim and Polyphase Slide Rule, Mr. Edmund D. Searles, New Bedford.

6. Special means for creating interest in the study of geometry, Miss Margaret English Bourne.

7. Geometry Understood, not Memorized, Rolland R. Smith, Newton.

THE Thirty-fourth Educational Conference of the Academies and High Schools in Relations with the University of Chicago was held May 11-12. The mathematics section met on Friday afternoon with Prof. H. E. Slaught presiding and the following topics were discussed:

1. A Study of the Size of Classes in Chicago High Schools, P. R. Stevenson, Ohio State University.

2. A Study of Mathematics under the Individual System, Miss Mary M. Reese, Horace Mann School, Winnetka, Ill.

3. The Teaching of Secondary Mathematics from the Point of View of the University, Prof. Ernest J. Wilczynski, University of Chicago.

4. Algebra Classes Graded according to Ability, Miss Lucy Price, New Trier Township School, Kenilworth, Ill.

Professor Stevenson gave a preliminary report of investigations based on a study of large and small classes built up in such a way as to have classes of equal ability as measured by intelligence tests. The semester average grades were almost the same in the large and small classes studied. In the discussion which followed it seemed to be the general opinion that factors other than size of classes had not been eliminated in making the study.

The second paper was a discussion of a method of individual work and promotion used in the Winnetka, Ill. grade schools. Definite standards of content and efficiency are planned and the pupil is promoted whenever he has reached these standards.

Professor Wilczynski stressed the point that calculation is not the main thing in mathematics, but the establishment of truths by logical argument. To illustrate this, he showed that no amount of calculation can reveal the nature of the square root of a number while a very brief logical argument will do so.

Miss Price showed the advantages of mathematics classes graded according to ability. In New Trier Township High School the classification was based upon intelligence tests given in the grade schools supplemented by teachers' estimates. The program in high school was so constructed as to make transition from one class to another easy. The pupil understood in which class, accelerated, normal, or slow, he was placed. Credit

is based upon the amount of work accomplished and not upon the amount of time spent in completing a given amount of work. (E. W. Owen, Oak Park, Ill.)

THE second annual meeting of the Inland Empire Council of Teachers of Mathematics was held at Spokane, April 5 and 6, in connection with the twenty-fourth annual session of the Inland Empire Teachers Association. The program consisted of an address on "Higher Mathematics as an Aid in High School Teaching" by Professor Eugene Taylor, of the University of Idaho; a report on "Correlated Mathematics in High School" by Miss Gertrude Kaye, of the North Central High School, Spokane; a report of the progress of the National Committee on Mathematical Requirements; and a round table symposium. The subjects discussed at this symposium were, "Why is election so small in third semester algebra," "What standard tests may be profitably used in secondary Mathematics," "What place should Mathetical Recreations have in High School Mathematics," and "The uses of the Slide Rule in High School Teaching."

Walter C. Eells, Professor of Applied Mathematics at Whitman College, was re-elected president of The Council, and Miss Olive Fisher, of the Lewis and Clark High School, Spokane, was elected Secretary. Professor Eugene Taylor was selected as chairman of a committee on Collegiate Mathematical Teaching, Miss Gertrude Kaye on Secondary School Mathematics, and Mr. W. H. Seale, chairman of a committee on Elementary School Mathematics. These committees are to make a special study of the report of the National Committee on Mathematical Requirements as related to the mathematical conditions of the North west states, Oregon, Washington, Idaho and Montana.

RESEARCH DEPARTMENT

This department has proposed two problems with the hope that data bearing on these two questions might be collected from many sources. The April (1922) issue proposed: "*To study the relation between general intelligence of pupils and achievement in a demonstrative geometry course.*" The second problem was stated as follows in the May (1922) issue: "*To study the relation between the general intelligence of pupils and achievement in a regular algebra course.*" A considerable amount of material has been received for each problem but the data secured are not as complete as they need to be to make possible a reliable study. The chief defect arises through the practice of sending material which neglects to account for pupils who *withdraw* before the close of the semester (term, or year). It may also prove desirable to indicate what pupils are *repeaters* and the number of times they have failed the course which they are attempting.

It seems urgent therefore to repeat the request for data. The material is identical for the two problems except for one part as indicated below.

What Data to Collect.—The readers interested in one or the other problems are asked to report:

1. One or more full class lists of pupils who *began* a demonstrative geometry course (or for the second problem a list of students who began a standard algebra course).

2. The final mark of each pupil in geometry for the first semester, the second semester or both (for the algebra problem report the marks in algebra). Indicate the standing of pupils who *withdrew*.

3. Report the I. Q. for each pupil listed in (1) and (2) or if this is unknown report the score (or scores) on one (or more) widely used group intelligence test, such as the Terman, Otis, National, and the like.

4. Indicate what pupils are repeating the course and the number of times they have previously failed the same course.

Summary Table.—It is hoped that data collected with the above directions may prove valuable to the readers of the MATH-

EMATICS TEACHER. The names of pupils will in no event be published. If a school prefers not to have its name published, that fact should be stated at the head of the report. The complete data should be reported not later than March 1, 1923.

RALEIGH SCHORLING,
The Lincoln School.

NEW BOOKS

Plane Geometry Review. By MURRAY J. LEVENTHAL and M. WEINER. Globe Book Company, New York. Pp. 86.

This review outline contains a statement of the basal propositions of plane geometry, suggestions for the proof of these propositions, numerical exercises, and a large number of recent College Entrance and Regents examination questions.

Geometry Note Book. Globe Book Company, New York.

The use of such a perforated tablet will systematize written proofs of propositions and "originals," and save considerable time for both student and teacher.

Standardized Reasoning Tests in Arithmetic and How to Utilize Them. Cliff W. Stone, Teachers' College Contributions to Education, No. 83; second edition, revised and enlarged.

This manual furnishes complete directions for giving the Stone Reasoning Tests, and for scoring the papers. Standards are given, and one chapter is devoted to setting forth effective means of representing the scores. Another chapter is devoted to a consideration of the utilization of the results of the test in the improvement of teaching.

This publication should be of use in supervision, and in the teaching of the reasoning phases of arithmetic.

Réflexions sur la Métaphysique du Calcul Infinitésimal. By Lazaire Carnot. Gauther-Villars et Cie., Paris, 1921, 2 vols. Vol. I, pp. viii + 118; vol II, pp. 106.

For those who are interested in the possibility of the introduction of the calculus into the senior high school and who feel that the pupil advances over somewhat the same route that the world has taken, these little paper-covered volumes are well worth reading. Indeed, those who teach the calculus in our colleges would not occupy their time amiss in finding how the subject stood at the close of the Napoleonic era, before Cauchy had made firm the foundations upon which the calculus has since rested. They will not find much assistance in the way of presenting the theory to their classes, but they will find a discussion of the philosophical principles which lie at the base of the discipline.

Lazare-Nicolas-Marguerite Carnot (1753-1823) was a man of great attainments. His early training gave him a substantial knowledge of the classics, of philosophy, and of mathematics. Like most young men of good family, in those days, he entered the army, and in the course of events he took a prominent part in the French Revolution, being honored by various political preferments. Apparently about the close of the old régime he wrote his *Réflexions* for in the first edition of the work "An V, (1797)" he says "Il y a quelques années que l'Auteur de ces Réflexions les a rédigées dans la forme où on les présente anjourd'hui." At the time of the publication of the essay he was too much occupied with his public duties to revise it, and so it appeared as a small volume of only eighty pages. During the Empire, however, he found time for writing, and so he published several noteworthy works upon modern geometry and revised and greatly extended the *Réflexions*, the second edition of the latter appearing in 1813. It is this second edition that is now republished in a series known as *Les Maitres de la Pensée Scientifique* and issued at a price within the reach of all French scholars.

The present edition is particularly interesting to the student of mathematical history in that it shows the working of Carnot's mind in seeking to relieve the calculus of the doubts that had prevailed in many instances from the time that Dean (afterwards Bishop) Berkley published *The Analyst* in 1734. The *ment petite* as "la différence de deux grandeurs qui ont pour limite une meme grandeur et rien de plus." This was not satisfactory, even in 1797, and it was less satisfactory after the critics of his book had done with their task. Hence, in 1813, Carnot rewrote that part of the essay that follows page 20 of the original edition, and took the most important step that had been taken up to that time to make the theory rigidly precise. That he did not succeed to the extent that Cauchy did is not to criticise his efforts; it is simply to say that Cauchy could not have accomplished what he did if he had not had Carnot's work on which to build.

A word of thanks is due to M. Solovine, the editor of the series, and to the well-known publishing house of Gauthier-Villars et Cie, for rendering such classics available and for reproducing them with such fidelity to the original text. They appear at

only about 3 francs a volume, in paper covers,—about 25 cents at the present rate of exchange. Unfortunately, the price in New York is likely to be much higher, for some reason that book-buyers cannot comprehend but that importers seem to feel that they can justify in spite of the fact that the books are admitted duty free and that the postage is only nominal.

DAVID EUGENE SMITH.

Mémoire sur la Chaieur. By M. Lavoisier and de Laplace, Paris, Gauthier-Villars et Cie, 1920, pp. 78 + 2 plates.

This memoir originally appeared in the *Mémories de l'Académie des Sciences*, in 1780, and is now republished in accord with the general plan of M. Solvine, the editor of the series "Les Maitres de la Pensée Scientifique." This plan is to make available for French students, at a nominal price, some of the great scientific classics which have appeared in their language, but which are now out of print. This particular memoir was written by two of the best-known scientists living in France at the period of the Revolution and it represents the enthusiasm of young manhood. An Edinburgh scientist, Joseph Black, had already worked upon the theory of latent and specific heat and had devised various calorimetric methods. Laplace, then only thirty-one years old, and Lavoisier, six years his senior, took up the problem and, in this work, they gave the results of their experiments. They set forth new methods of measuring heat and applied these methods to the determination of the specific heat of various substances as related to that of water taken as a standard. They then considered the general theory of heat as based upon these observations.

The work is not one of special importance to the teacher of mathematics, but to the physicist it will be interesting and profitable reading.

DAVID EUGENE SMITH.